Lowest Landau Level Bosonization

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We develop a bosonization scheme for the two-dimensional electron gas in the presence of an uniform magnetic field perpendicular to the two-dimensional system when the filling factor is one ($\nu = 1$). We show that the elementary neutral excitations of this system, known as magnetic excitons, can be treated approximately as bosons and we apply the method to the interacting system. We show that the Hamiltonian of the fermionic system is mapped into an interacting bosonic Hamiltonian, where the dispersion relation of the bosons agrees with previous calculations of Kallin and Halperin. The interaction term accounts for the formation of bound states of two-bosons. We discuss a possible relation between these excitations and the skyrmion-antiskyrmion pair, in analogy with the ferromagnetic Heisenberg model. Finally, we analyze the semiclassical limit of the interacting bosonic Hamiltonian and show that the results are in agreement with those derived from the model of Sondhi *et al.* for the quantum Hall skyrmion.

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I. INTRODUCTION

Bosonization of fermionic systems is a nonperturbative method which has become a very useful tool for treating strongly correlated systems in one-dimension. The basic idea of this approach consists of describing the neutral elementary excitations of the system in a bosonic language, which allows us to map the sometimes intractable fermionic system into a more friendly bosonic model. A very detailed description of the so-called constructive one-dimensional bosonization method, its relations with the field-theoretical approach and some applications can be found in[1,2] and the references therein. Some efforts have also been made to extend this formalism to higher dimensions. The first attempt was carried out by Luther[3] and extended by Haldane.[4] A bosonization method for a Fermi liquid in any number of dimensions was constructed by Castro Neto and Fradkin[5] and also by Houghton and Marston.[6,7]

The quantum Hall effect is one of the most interesting phenomenon observed in the two-dimensional electron systems (for a review see Refs.[8,9,10,11]). While the integral quantum Hall effect can be understood in terms of a noninteracting electron model, correlation effects due to the Coulomb interaction between the electrons are important to understand the fractional quantum Hall effect. An exception to the above scenario is the quantum Hall system at filling factor one ($\nu = 1$), where the electron-electron interaction also plays an important role.

A bosonization approach for the two-dimensional electron gas subject to an external perpendicular magnetic field was developed by Westfahl Jr. *et al.*[12] In this case, it was shown that the elementary neutral excitations of the system, known as magnetic excitons[13], can be described in a bosonic language and the Hamiltonian of the interacting two-dimensional electron (2DEG) gas was mapped into a quadratic bosonic Hamiltonian. However, this method can be applied only in the limit of small external field when a large (integer) number of Landau levels are completely filled. A different bosonization scheme for the collective dynamics of a spinless 2DEG in the lowest Landau level was developed by Conti and Vignale.[15]

As pointed out above, the two-dimensional electron gas at $\nu = 1$ is a strongly correlated electron system. It is well known that the ground state of this system is a spin-polarized state in which all electrons completely fill the lowest Landau level with spin up polarization (quantum Hall ferromagnet).[8,9,10,11] The elementary neutral excitations are also described as magnetic excitons[13] which, in the long wavelength limit, can be seen as spin wave excitations of the the quantum Hall ferromagnet. Moreover, the low lying charged excitation is described by a charged spin texture, called quantum Hall skyrmion.[16] This non trivial excitation can be viewed as a configuration in which the spin field at the center points down and then it rotates smoothly as we move radially outwards from the center until all the spins point up as in the ground state. These charged spin textures are topologically stable objects with size (the number of reversed spins) fixed by the competition between the Coulomb and Zeeman interactions.

Since the quantum Hall system at $\nu = 1$ is a quite interesting strongly correlated electron system and the bosonization was successful in describing the integral quantum Hall system at $\nu \gg 1$, we would like to extend the methodology developed by Westfhal Jr *et al.*[12] to the case when the system is under the effect of a high external magnetic field, in particular, when the filling factor is one. This is precisely the aim of this paper.

Following the ideas of Ref. 12 and 17, we start with a

Landau level description of this system and then we introduce a nonperturbative bosonization approach for it. We follow Tomonaga's ideas for the one-dimensional electron gas in order to show that the neutral excitations of 2DEG, the electron-hole pairs called *magnetic excitons*, can be treated approximately as bosons.

We will show that the Hamiltonian of the interacting two-dimensional electron gas at $\nu = 1$ can be mapped into an interacting bosonic Hamiltonian, where the single particle energy of the bosons is equal to the energy of the magnetic excitons derived by Kallin and Halperin.[13] The interaction between the bosons gives rise to the formation of two-boson bound-states. In analogy with the isotropic Heisenberg model, these bound states can be related to the skyrmion-antiskyrmion pair, which is also a neutral excitation of the system.

The paper is organized as follows. In Sec. II, we will present the bosonization scheme for the 2DEG at $\nu = 1$, i.e, the bosonic operators will be defined. The bosonic representation of the density and spin operators will be derived and we will show the relation between the lowest Landau Level (LLL) projection formalism and the bosonization method; the reorganization of the Hilbert space will be also discussed. In Sec. III, we will apply the method to study the interacting two-dimensional electron gas at $\nu = 1$. Finally, in Sec. IV, we will analyze the semiclassical limit of the interacting bosonic Hamiltonian derived in the previous section following the procedure of Moon *et al.*[18]

II. THE BOSONIZATION METHOD FOR THE 2DEG AT $\nu = 1$

A. bosonic operator definition

Let us consider N spinless noninteracting electrons moving in the $x-y$ plane (two-dimensional electron gas) in an external field $\mathbf{B} = B_0\hat{z}$. The system is described by the Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2m^*}\int d^2r\ \Psi^\dagger(\mathbf{r})\left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2\Psi(\mathbf{r}), \quad (1)$$

where m^* is the effective mass of the electron in the host semiconductor (see Appendix A) and $\Psi^\dagger(\mathbf{r})$ is the fermionic field operator. In the symmetric gauge, the vector potential $\mathbf{A}(\mathbf{r}) = -(\mathbf{r}\times\mathbf{B})/2$ and therefore the fermionic field operators can be written in a Landau level basis as (Appendix B)

$$\begin{aligned}
\Psi^\dagger(\mathbf{r}) &= \sum_{n,m}\langle n\ m|\mathbf{r}\rangle c^\dagger_{n\,m} \\
&= \sum_{n,m}\frac{1}{\sqrt{2\pi l^2}}e^{-|r|^2/4l^2}G^*_{m+n,n}(ir/l)c^\dagger_{n\,m}, \\
\Psi(\mathbf{r}) &= \sum_{n,m}\langle\mathbf{r}|n\ m\rangle c_{n\,m} \qquad (2)\\
&= \sum_{n,m}\frac{1}{\sqrt{2\pi l^2}}e^{-|r|^2/4l^2}G_{m+n,n}(ir/l)c_{n\,m},
\end{aligned}$$

where $r = x+iy$, $l = \sqrt{\hbar c/(eB)}$ is the magnetic length and the function $G_{m+n,n}(x)$ is defined in Appendix C. The fermionic operator $c^\dagger_{n\,m}$ creates an electron in the Landau level n with guiding center m and obeys canonical anti-commutation relations

$$\begin{aligned}
\{c^\dagger_{n\,m}, c^\dagger_{n'\,m'}\} &= \{c_{n\,m}, c_{n'\,m'}\} = 0 \\
\{c^\dagger_{n\,m}, c_{n'\,m'}\} &= \delta_{n\,n'}\delta_{m\,m'},
\end{aligned} \qquad (3)$$

with $n = 0,1,2,\ldots$ and $m = 0,1,\ldots,N_\phi - 1$. Here, $N_\phi = \mathcal{A}n_B$ is the degeneracy of each Landau level, with $n_B = 1/(2\pi l^2)$ and $\mathcal{A}$ is the area of the system. Substituting Eqs. (2) in Eq. (1), we find that the Hamiltonian of the system is diagonal in the Landau level basis,

$$\mathcal{H}_0 = \sum_{n,m}\hbar w_c(n+\frac{1}{2})c^\dagger_{n\,m}c_{n\,m}, \qquad (4)$$

where $w_c = eB/(m^*c)$ is the cyclotron frequency. Defining the filling factor $\nu = N/N_\phi$ as the number of filled Landau levels, for ν integer, the ground state of the two-dimensional electron gas (2DEG) is obtained by completely filling the ν lowest Landau levels,

$$|GS\rangle = \prod_{n=0}^{\nu-1}\prod_{m=0}^{N_\phi-1}c^\dagger_{n\,m}|0\rangle \qquad (5)$$

where $|0\rangle$ is the vacuum state.

Now, if we consider the electronic spin and restrict the Hilbert space to the lowest Landau level ($n = 0$) only, the fermionic field operators (2) become

$$\begin{aligned}
\Psi^\dagger_\sigma(\mathbf{r}) &= \sum_m\frac{1}{\sqrt{2\pi l^2}}e^{-|r|^2/4l^2}G_{0,m}(-ir^*/l)c^\dagger_{m\,\sigma}, \\
\Psi_\sigma(\mathbf{r}) &= \sum_m\frac{1}{\sqrt{2\pi l^2}}e^{-|r|^2/4l^2}G_{m,0}(ir/l)c_{m\,\sigma},
\end{aligned} \qquad (6)$$

where $c^\dagger_{m\,\sigma}$ creates a spin σ electron in the lowest Landau level with guiding center m. These creation and annihilation fermionic operators also obey the anticommutation relations,

$$\begin{aligned}
\{c^\dagger_{m\,\sigma}, c^\dagger_{m'\,\sigma'}\} &= \{c_{m\,\sigma}, c_{m'\,\sigma'}\} = 0, \\
\{c^\dagger_{m\,\sigma}, c_{m'\,\sigma'}\} &= \delta_{m,m'}\delta_{\sigma,\sigma'},
\end{aligned} \qquad (7)$$

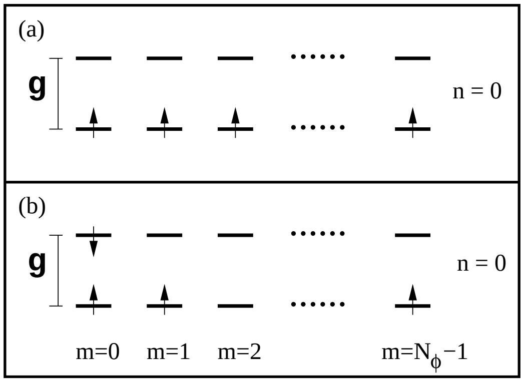

FIG. 1: Schematic representation of (a) the 2DEG ground state at $\nu = 1$ (quantum Hall ferromagnet) and (b) the elementary neutral excitation (the electron-hole pair). g is the Zeeman energy.

with $\sigma =\uparrow$ or $\downarrow$.

In addition to the kinetic energy term $\mathcal{H}_0$ [Eq. (1)], we should also include a Zeeman energy term $\mathcal{H}_\mathcal{Z}$ in the total Hamiltonian of the system,

$$\mathcal{H}_\mathcal{Z} = -\frac{1}{2} g^* \mu_B B \sum_\sigma \int d^2r \; \sigma \Psi^\dagger_\sigma(\mathbf{r}) \Psi_\sigma(\mathbf{r}), \qquad (8)$$

where $g^* > 0$ is the effective electron g-factor (see Appendix A) and μ_B is the Bohr magneton. Substituting Eqs. (6) in the expressions (1) and (8), the total Hamiltonian of the 2DEG, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\mathcal{Z}$, yields

$$\mathcal{H} = \frac{1}{2} \sum_{m,\sigma} \hbar w_c c^\dagger_{m\,\sigma} c_{m\,\sigma} - \frac{1}{2} g^* \mu_B B \sum_{m,\sigma} \sigma c^\dagger_{m\,\sigma} c_{m\,\sigma}. \quad (9)$$

We can see that $\mathcal{H}$ is also diagonal in the Landau level basis and the kinetic energy term is simply a constant. The one particle energy eigenvalues are $-g^*\mu_B/2$ and $g^*\mu_B/2$ whereas the degeneracy of each energy eigenstates is N_ϕ.

For $\nu = 1$, the ground state of the 2DEG, $|FM\rangle$, is obtained by completely filling the spin up lowest Landau level (the quantum Hall ferromagnet)

$$|FM\rangle = \prod_{m=0}^{N_\phi - 1} c^\dagger_{m\,\uparrow} |0\rangle, \qquad (10)$$

as illustrated in Fig. 1(a). In this case, each guiding center is occupied by only one electron with spin up. $|FM\rangle$ is an eigenvector of the operator S_z (the z component of the total spin) whose eigenvalue is equal to $N_\phi/2$.

The neutral elementary excitations of the system are electron-hole pairs or spin flips as one spin up electron is annihilated and one spin down electron is created in the quantum Hall ferromagnet [Fig. 1.(b)]. These excited states $|\Psi\rangle$ can be constructed by acting with the spin operator $S^- = S_x - iS_y$ on the ground state $|FM\rangle$,

$$|\Psi\rangle \propto S^- |FM\rangle.$$

In the bosonization approach for the one-dimensional electron gas, the annihilation and creation bosonic operators are derived from the electron density operator $\hat{\rho}(k)$ as the electron-hole pair excitations can be obtained by acting with $\hat{\rho}(k)$ on the ground state of the system[1,2]. In order to define the bosonic operators, the commutation relation between the electron density operators $\hat{\rho}(k)$ with different momenta is analyzed. We will follow the same procedure, but here we will analyze the spin density operators $S^+(\mathbf{r}) = S_x(\mathbf{r}) + iS_y(\mathbf{r})$ and $S^-(\mathbf{r}) = S_x(\mathbf{r}) - iS_y(\mathbf{r})$ in order to define the bosonic operators for the 2DEG at $\nu = 1$. More precisely, we are interested in the Fourier transform of these spin operators.

Before doing that, we need to say some words about the density operator of spin σ electrons, which is defined as

$$\hat{\rho}_\sigma(\mathbf{r}) = \Psi^\dagger_\sigma(\mathbf{r}) \Psi_\sigma(\mathbf{r}).$$

With the aid of the expressions (6) and the definition of the function $G_{m,m'}(x)$ (Appendix C), it is possible to show that

$$\begin{aligned}
\hat{\rho}_\sigma(\mathbf{q}) &= \int d^2r \; e^{-i\mathbf{q}\cdot\mathbf{r}} \Psi^\dagger_\sigma(\mathbf{r}) \Psi_\sigma(\mathbf{r}) \\
&= \sum_{m,m'} \int d^2r \; e^{-i\mathbf{q}\cdot\mathbf{r}} \langle m|\mathbf{r}\rangle\langle\mathbf{r}|m'\rangle c^\dagger_{m\,\sigma} c_{m'\,\sigma} \\
&= \sum_{m,m'} \langle m| e^{-i(qz^\dagger + q^* z)/2} |m'\rangle c^\dagger_{m\,\sigma} c_{m'\,\sigma} \\
&= e^{-|lq|^2/2} \sum_{m,m'} G_{m,m'}(lq) c^\dagger_{m\,\sigma} c_{m'\,\sigma}, \qquad (11)
\end{aligned}$$

where $q = q_x + iq_y$ and the operator z is defined by the Eq. (C1). Observe that the action of $\hat{\rho}_\sigma(\mathbf{q})$ on $|FM\rangle$ *does not* create any electron-hole pair excitations.

The spin density operator is defined as ($\hbar = 1$)

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2} \sum_{\alpha,\beta} \Psi^\dagger_\alpha(\mathbf{r}) \hat{\sigma}_{\alpha,\beta} \Psi_\beta(\mathbf{r}),$$

where the components of the vector $\hat{\sigma}$ are the Pauli matrices. However, we will define the spin density operators $S^+(\mathbf{r})$ and $S^-(\mathbf{r})$ only by the nonzero matrix element. In terms of the fermionic field operators we have

$$\begin{aligned}
S^+(\mathbf{r}) &\equiv \Psi^\dagger_\uparrow(\mathbf{r}) \Psi_\downarrow(\mathbf{r}), \\
S^-(\mathbf{r}) &\equiv \Psi^\dagger_\downarrow(\mathbf{r}) \Psi_\uparrow(\mathbf{r}).
\end{aligned}$$

In analogy with equation (11), it is easy to show that the Fourier transform of these spin operators is given by

$$S^+(\mathbf{q}) = e^{-|lq|^2/2} \sum_{m,m'} G_{m,m'}(lq) c^\dagger_{m\,\uparrow} c_{m'\,\downarrow} \qquad (12)$$

$$S^-(\mathbf{q}) = e^{-|lq|^2/2} \sum_{m,m'} G_{m,m'}(lq) c^\dagger_{m\,\downarrow} c_{m'\,\uparrow}. \qquad (13)$$

After some algebra, it is possible to show that the commutation relation between the operators $S^{+}(\mathbf{q})$ and $S^{-}(\mathbf{q})$ is proportional to the Fourier transform of the density operators $\hat{\rho}_{\uparrow}(\mathbf{r})$ and $\hat{\rho}_{\downarrow}(\mathbf{r})$ (see Appendix D),

$$[S^{+}_{\mathbf{q}}, S^{-}_{\mathbf{q}'}] \;=\; e^{l^2 q q'^{*}/2}\hat{\rho}_{\uparrow}(\mathbf{q}+\mathbf{q}') - e^{l^2 q' q^{*}/2}\hat{\rho}_{\downarrow}(\mathbf{q}+\mathbf{q}').$$

Now, as the average values of $\hat{\rho}_{\uparrow}(\mathbf{q})$ and $\hat{\rho}_{\downarrow}(\mathbf{q})$ in the ground state (10) are $\langle\hat{\rho}_{\uparrow}(\mathbf{q}+\mathbf{q}')\rangle = N_{\phi}\delta_{\mathbf{q}+\mathbf{q}',0}$ and $\langle\hat{\rho}_{\downarrow}(\mathbf{q}+\mathbf{q}')\rangle = 0$, respectively, the average value of the commutator $[S^{+}_{\mathbf{q}}, S^{-}_{\mathbf{q}'}]$ is

$$\langle[S^{+}_{\mathbf{q}}, S^{-}_{\mathbf{q}'}]\rangle = e^{-|lq|^2/2}N_{\phi}\delta_{\mathbf{q}+\mathbf{q}',0}. \tag{14}$$

The above expression will allow us to define the bosonic operators as a function of the fermionic operators $c^{\dagger}_{m\,\sigma}$ and $c_{m\,\sigma}$.

If we define the operators $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$ by

$$b_{\mathbf{q}} \;\equiv\; \frac{1}{\sqrt{N_{\phi}}}e^{|lq|^2/4}S^{+}_{-\mathbf{q}} \tag{15}$$

$$b^{\dagger}_{\mathbf{q}} \;\equiv\; \frac{1}{\sqrt{N_{\phi}}}e^{|lq|^2/4}S^{-}_{\mathbf{q}}, \tag{16}$$

and if we approximate the commutation relation between the $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$ by its average value in the ground state (10),

$$[b_{\mathbf{q}}, b^{\dagger}_{\mathbf{q}'}] \approx \langle[b_{\mathbf{q}}, b^{\dagger}_{\mathbf{q}'}]\rangle = \delta_{\mathbf{q},\mathbf{q}'}, \tag{17}$$

we can say that $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$ are approximately bosonic operators. In analogy with the Tomonaga's model for one-dimensional electron gas, we will assume that (17) is exact.[19,20] This is the main approximation of our method. Observe that this approximation is quite similar to the one adopted in the random phase approximation.[21]

Therefore, after this point, we will assume that

$$b_{\mathbf{q}} \;=\; \frac{1}{\sqrt{N_{\phi}}}e^{-|lq|^2/4}\sum_{m,m'}G_{m,m'}(-lq)c^{\dagger}_{m\,\uparrow}c_{m'\,\downarrow} \tag{18}$$

$$b^{\dagger}_{\mathbf{q}} \;=\; \frac{1}{\sqrt{N_{\phi}}}e^{-|lq|^2/4}\sum_{m,m'}G_{m,m'}(lq)c^{\dagger}_{m\,\downarrow}c_{m'\,\uparrow}, \tag{19}$$

are bosonic operators, which obey the canonical commutation relations

$$\begin{aligned}[][b^{\dagger}_{\mathbf{q}}, b^{\dagger}_{\mathbf{q}'}] \;&=\; [b_{\mathbf{q}}, b_{\mathbf{q}'}] = 0, \\ [b_{\mathbf{q}}, b^{\dagger}_{\mathbf{q}'}] \;&=\; \delta_{\mathbf{q},\mathbf{q}'}.\end{aligned} \tag{20}$$

The quantum Hall ferromagnet $|FM\rangle$ is the vacuum state for the bosons as the action of the fermionic operator $c_{m'\,\downarrow}$ on $|FM\rangle$ is equal to zero. Therefore the bosonic Hilbert space is spanned by applying the operator $b^{\dagger}_{\mathbf{q}}$ on the quantum Hall ferromagnet any number of times

$$|\{n_{\mathbf{q}}\}\rangle = \prod_{q\in\{n_{\mathbf{q}}\}}\frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}|0\rangle = \prod_{\mathbf{q}}\frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}|FM\rangle, \tag{21}$$

with $n_{\mathbf{q}} \geq 0$ and $\sum n_{\mathbf{q}} \leq N_{\phi}$.

The state $b^{\dagger}_{\mathbf{q}}|FM\rangle$ is a linear combination of electron-hole excited states as illustrated in Fig. 1(b), where both the electron and the hole have spin down. In fact, the bosonic operator $b^{\dagger}_{\mathbf{q}}$ is similar to the operator $e^{\dagger}_{n,p}(q)$, with $n = p = 0$, discussed in Ref. 12. This operator creates the neutral excitations known as magnetic excitons when it is applied on the ground state of the noninteracting two-dimensional electron gas [Eq.(5)].

The momentum $\mathbf{q}$ is canonically conjugate to the vector $\mathbf{R}_0 = (\mathbf{R}^{e}_0 + \mathbf{R}^{h}_0)/2$. Here, the vectors $\mathbf{R}^{e}_0$ and $\mathbf{R}^{h}_0$ are, respectively, the position of the guiding centers of the electron and the hole excited in the system as defined in Appendix B. Hence $\mathbf{R}$ is the center of mass of the guiding centers of the excited electron and hole. In addition, the momentum $\mathbf{q}$ is a good quantum number because the total momentum of a two-dimensional system of charged particles in an external magnetic field is conserved when the total charge of the system is zero.[22]

To sum up, we can say that the state $b^{\dagger}_{\mathbf{q}}|FM\rangle$ is a neutral elementary excitation of the 2DEG at $\nu = 1$ which corresponds to either a spin-flip or a magnetic exciton with momentum $\mathbf{q}$.

B. Density operator

Although the bosonic operators are not directly derived from the electron density operator as in the one-dimensional electron gas, the latter is a very useful operator when the Coulomb interaction between the electrons of the 2DEG is considered. In this section, we will show that it is possible to write down the electron density operator as a product of the bosonic operators $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$.

The bosonic representation of any operator $\mathcal{O} = \mathcal{O}(c^{\dagger}_{m,\sigma}, c_{m',\sigma})$ can be obtained applying this operator on the eigenstates (21), which span the bosonic Hilbert space,

$$\begin{aligned}\mathcal{O}|\{n_{\mathbf{q}}\}\rangle \;&=\; \mathcal{O}\left(\prod_{\mathbf{q}}\frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}\right)|FM\rangle \\ &=\; [\mathcal{O}, \prod_{\mathbf{q}}\frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}]|FM\rangle + \prod_{\mathbf{q}}\frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}\mathcal{O}|FM\rangle.\end{aligned} \tag{22}$$

Starting with the expressions of $\mathcal{O}$ and $b^{\dagger}_{\mathbf{q}}$ as a function of the fermionic operators $c^{\dagger}_{m\,\sigma}$ and $c_{m'\,\sigma}$, we can calculate the value of $\mathcal{O}|FM\rangle$ as well as the commutation relation $[\mathcal{O}, b^{\dagger}_{\mathbf{q}}]$, which allows us to obtain the value of the commutator in Eq.(22).

Following the above procedure, let us derive the expression of the density operator of spin up electrons $\hat{\rho}_{\uparrow}(\mathbf{k})$ as a function of the b's. It is quite easy to show that

$$[\hat{\rho}_{\uparrow}(\mathbf{k}), b^{\dagger}_{\mathbf{q}}] \;=\; -e^{-|lk|^2/4}e^{-i\mathbf{k}\wedge\mathbf{q}/2}b^{\dagger}_{\mathbf{k}+\mathbf{q}},$$

where $\mathbf{k}\wedge\mathbf{q} = l^2\hat{z}\cdot(\mathbf{k}\times\mathbf{q})$. Using the property (C12) of the function $G_{m,m'}(x)$, we can see that the value of

$\hat{\rho}_{\uparrow}(\mathbf{k})|FM\rangle$ is simply a constant,

$$\hat{\rho}_{\uparrow}(\mathbf{k})|FM\rangle = e^{-|lk|^2/2}\sum_{m,m'} G_{m,m'}(lk)c^{\dagger}_{m\uparrow}c_{m'\uparrow}|FM\rangle$$
$$= \delta_{\mathbf{k},0}N_{\phi}|FM\rangle. \quad (23)$$

After some algebra, we end up with

$$\hat{\rho}_{\uparrow}(\mathbf{k})|\{n_{\mathbf{q}}\}\rangle = -\sum_{p\in\{n_{\mathbf{q}}\}} e^{-|lk|^2/4 - i\mathbf{k}\wedge\mathbf{p}/2} b^{\dagger}_{\mathbf{k}+\mathbf{p}} \frac{n_{\mathbf{p}}(b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}-1}}{\sqrt{n_{\mathbf{p}}!}}$$
$$\times \prod_{q\in\{n_{\mathbf{q}}\},q\neq p} \frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}|FM\rangle$$
$$+\ \delta_{\mathbf{k},0}N_{\phi}|\{n_{\mathbf{q}}\}\rangle$$
$$= -e^{-|lk|^2/4}\sum_{\mathbf{p}} e^{-i\mathbf{k}\wedge\mathbf{p}/2} b^{\dagger}_{\mathbf{k}+\mathbf{p}} b_{\mathbf{p}}|\{n_{\mathbf{q}}\}\rangle$$
$$+\delta_{\mathbf{q},0}N_{\phi}|\{n_{\mathbf{q}}\}\rangle. \quad (24)$$

In the second step above, we used the fact that

$$b_{\mathbf{p}}(b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}} = [b_{\mathbf{p}},(b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}}] + (b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}}b_{\mathbf{p}}$$
$$= n_{\mathbf{p}}(b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}-1} + (b^{\dagger}_{\mathbf{p}})^{n_{\mathbf{p}}}b_{\mathbf{p}}$$

and that $b_{\mathbf{p}}|FM\rangle = 0$. As Eq. (24) is valid for any eigenstate of the bosonic Hilbert space, i.e., it is an operator identity, we can conclude that

$$\hat{\rho}_{\uparrow}(\mathbf{k}) = \delta_{\mathbf{k},0}N_{\phi} - e^{-|lk|^2/4}\sum_{\mathbf{q}} e^{-i\mathbf{k}\wedge\mathbf{q}/2} b^{\dagger}_{\mathbf{k}+\mathbf{q}} b_{\mathbf{q}} \quad (25)$$

is the bosonic representation of the density operator $\hat{\rho}_{\uparrow}(\mathbf{k})$.

In the same way, the expression of the operator $\hat{\rho}_{\downarrow}(\mathbf{k})$ in terms of the b's is given by

$$\hat{\rho}_{\downarrow}(\mathbf{k}) = e^{-|lk|^2/4}\sum_{\mathbf{q}} e^{+i\mathbf{k}\wedge\mathbf{q}/2} b^{\dagger}_{\mathbf{k}+\mathbf{q}} b_{\mathbf{q}}, \quad (26)$$

as $\hat{\rho}_{\downarrow}(\mathbf{k})|FM\rangle = 0$ and

$$[\hat{\rho}_{\downarrow}(\mathbf{k}), b^{\dagger}_{\mathbf{q}}] = e^{-|lk|^2/4} e^{+i\mathbf{k}\wedge\mathbf{q}/2} b^{\dagger}_{\mathbf{k}+\mathbf{q}}.$$

An alternative procedure to obtain Eqs. (25) and (26) consists of looking for an expression in terms of the b's which satisfies the commutation relation $[\mathcal{O}, b^{\dagger}_{\mathbf{q}}]$. For instance, for the electron density operator $\hat{\rho}_{\uparrow}(\mathbf{k})$, the commutator $[\hat{\rho}_{\uparrow}(\mathbf{k}), b^{\dagger}_{\mathbf{q}}] \propto b^{\dagger}_{\mathbf{q}+\mathbf{k}}$ and therefore we can conclude that the expression of $\hat{\rho}_{\uparrow}(\mathbf{k})$ in terms of the b's should be a linear combination of the operator $b^{\dagger}_{\mathbf{q}+\mathbf{k}}b_{\mathbf{q}}$. Choosing the coefficients properly, we easily find the first term of Eq.(25). In order to obtain the complete expression, it is necessary to add the term related to the action of $\hat{\rho}_{\uparrow}(\mathbf{k})$ on $|FM\rangle$. In the next sections, we will adopt this procedure as it is simpler than the one previously discussed.

Finally, from equations (25) and (26), we arrive at the bosonic form of the density operator

$$\hat{\rho}_{\mathbf{k}} = \hat{\rho}_{\uparrow}(\mathbf{k}) + \hat{\rho}_{\downarrow}(\mathbf{k}) \quad (27)$$
$$= \delta_{\mathbf{k},0}N_{\phi} + 2ie^{-|lk|^2/4}\sum_{\mathbf{q}} \sin(\mathbf{k}\wedge\mathbf{q}/2) b^{\dagger}_{\mathbf{k}+\mathbf{q}} b_{\mathbf{q}}.$$

Notice that $\hat{\rho}_{\mathbf{k}}$ is quadratic in the bosonic operators.

C. Spin density operators

In this section, we will derive the bosonic representation of the spin operators $S^z_{\mathbf{k}}$, $S^+_{\mathbf{k}}$ and $S^-_{\mathbf{k}}$. We will see that the obtained forms for these operators are similar to the ones of the formalism suggested by Dyson to study spin waves in a ferromagnetic system.[23]

Since the z-component of the spin density operator can be defined as

$$S^z(\mathbf{r}) = \frac{1}{2}\left(\Psi^{\dagger}_{\uparrow}(\mathbf{r})\Psi_{\uparrow}(\mathbf{r}) - \Psi^{\dagger}_{\downarrow}(\mathbf{r})\Psi_{\downarrow}(\mathbf{r})\right),$$

the expression of the Fourier transform of this operator as a function of the bosonic operators follows immediately from Eqs. (25) and (26),

$$S^z_{\mathbf{k}} = \frac{1}{2}\left(\rho_{\uparrow}(\mathbf{k}) - \rho_{\downarrow}(\mathbf{k})\right) \quad (28)$$
$$= \frac{1}{2}\delta_{\mathbf{k},0}N_{\phi} - e^{-|lk|^2/4}\sum_{\mathbf{q}} \cos(\mathbf{k}\wedge\mathbf{q}/2) b^{\dagger}_{\mathbf{k}+\mathbf{q}} b_{\mathbf{q}}.$$

In the spite of defining the bosonic operators $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$ from equations (15) and (16) respectively, the latter do not correspond to the bosonic representation of $S^+_{\mathbf{k}}$ and $S^-_{\mathbf{k}}$ as

$$[S^+_{\mathbf{q}}, S^-_{\mathbf{q}'}] \neq \delta_{\mathbf{q},\mathbf{q}'}.$$

In fact, we should also follow the procedure described in Sec.II B in order to calculate the bosonic form of the operators $S^+_{\mathbf{k}}$ and $S^-_{\mathbf{k}}$.

From equations (13) and (19) it is possible to show that $[S^-_{\mathbf{k}}, b^{\dagger}_{\mathbf{q}}] = 0$. Therefore, the action of $S^-_{\mathbf{k}}$ on the eigenstates (21) is related to the action of this operator on the quantum Hall ferromagnet $|FM\rangle$ only

$$S^-_{\mathbf{k}}|\{n_{\mathbf{q}}\}\rangle = \underbrace{[S^-_{\mathbf{k}}, \prod_{q\in\{n_{\mathbf{q}}\}} \frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}]}_{0}|FM\rangle$$
$$+ \prod_{q\in\{n_{\mathbf{q}}\}} \frac{(b^{\dagger}_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}} S^-_{\mathbf{k}}|FM\rangle.$$

Unlike the results for the electron density operator, $S^-_{\mathbf{k}}|FM\rangle$ is not just a constant, but rather

proportional to a linear combination of the terms $G_{m,m'}(lq)c^\dagger_{m\downarrow}c_{m'\uparrow}|FM\rangle$. In this case, it seems quite reasonable to consider Eq.(16) and define the bosonic representation of the operator $S^-_{\mathbf{k}}$ as

$$S^-_{\mathbf{k}} \equiv \sqrt{N_\phi}e^{-|lk|^2/4}b^\dagger_{\mathbf{k}}. \tag{29}$$

In the next section, we will show that (29) is very well defined because it satisfies the commutation relations between the spin density operators as well as between the spin density and electron density operators.

On the other hand, $S^+_{\mathbf{k}}|FM\rangle = 0$ [see Eq.(12)] and hence the bosonic representation of the operator $S^+_{\mathbf{k}}$ is completely determined by the commutation relation between this spin operator and $b^\dagger_{\mathbf{q}}$,

$$\begin{aligned}[][S^+_{\mathbf{k}}, b^\dagger_{\mathbf{q}}] &= \frac{e^{|lq|^2/4}}{\sqrt{N_\phi}}\left(e^{l^2kq^*/2}\rho_\uparrow(\mathbf{k}+\mathbf{q}) - e^{l^2k^*q/2}\rho_\downarrow(\mathbf{k}+\mathbf{q})\right) \\ &= \sqrt{N_\phi}e^{-|lk|^2/4}\delta_{\mathbf{k},-\mathbf{q}} - \frac{2}{\sqrt{N_\phi}}e^{-|lk|^2/4}\sum_{\mathbf{p}}\cos\left(\frac{\mathbf{k}\wedge(\mathbf{q}-\mathbf{p}) - \mathbf{q}\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{p}+\mathbf{q}}b_{\mathbf{p}}. \end{aligned} \tag{30}$$

As the first term of Eq.(30) is proportional to $\delta_{\mathbf{k},-\mathbf{q}}$, we can conclude that the operator $S^+_{\mathbf{k}}$ should present a term linear in $b_{\mathbf{k}}$. Moreover, the second term is a linear combination of products of the form $b^\dagger_{\mathbf{k}+\mathbf{p}}b_{\mathbf{p}}$, which implies that $S^+_{\mathbf{k}}$ should have a term proportional to the product $b^\dagger bb$. Choosing the coefficients properly, we end up with the bosonic form of the operator $S^+_{\mathbf{k}}$

$$S^+_{\mathbf{k}} = \sqrt{N_\phi}e^{-|lk|^2/4}b_{-\mathbf{k}} - \frac{1}{\sqrt{N_\phi}}e^{-|lk|^2/4}\sum_{\mathbf{p},\mathbf{q}}\cos\left(\frac{\mathbf{k}\wedge(\mathbf{q}-\mathbf{p}) - \mathbf{q}\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}}b_{\mathbf{q}}, \tag{31}$$

which satisfies the commutation relation (30).

As pointed out earlier, the representation (28), (29) and (31) is similar to the one previously considered by Dyson.[23] An important point of this formalism is that, althougth the Hermiticity requirement $S^+_{\mathbf{k}} = (S^-_{-\mathbf{k}})^\dagger$ does not hold, the usual commutation relations between the spin operators are satisfied. A detailed review of this formalism can be found in Ref. 24. As we will show in the next section, the representation (28), (29) and (31) derived using the bosonization method also preserves the commutation relation between the spin density operators.

D. LLL projection

In this section, we will show that, using the bosonic representation of the operators $\hat{\rho}_{\mathbf{k}}$, $S^z_{\mathbf{k}}$, $S^+_{\mathbf{k}}$ and $S^-_{\mathbf{k}}$, the commutation relations between them are in agreement with the results derived from the formalism of the Lowest Landau Level (LLL) projection.

The LLL projection is a formulation of the quantum mechanics in the restricted subspace formed by the lowest Landau level as developed by Girvin and Jach[25] (a brief review of this formalism is presented in Refs.[8,10]). An important consequence of the projection of the electron density and spin density operators on the LLL subspace is that the commutation relations between those operators are modified, i.e., the projected spin operators do not commute with the electron density operator and do not follow the canonical commutation relations between spin operators either.

From equation (27), it is quite easy to show that the commutation relation between electron density operators with distinct momenta is given by

$$\begin{aligned}[][\hat{\rho}_{\mathbf{k}}, \hat{\rho}_{\mathbf{q}}] &= 4e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\mathbf{q}\wedge\mathbf{k}/2\right)e^{-|lk+lq|^2/4} \\ &\quad\times\sum_{\mathbf{p}}\sin\left(\frac{(\mathbf{k}+\mathbf{q})\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}}. \end{aligned} \tag{32}$$

We can see that $[\hat{\rho}_{\mathbf{k}}, \hat{\rho}_{\mathbf{q}}]$ is proportional to a linear combination of the product $b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}}$ with coefficients equal to $\sin((\mathbf{k}+\mathbf{q})\wedge\mathbf{p}/2)$. This result indicates that the commutator should be related to the electron density operator $\hat{\rho}_{\mathbf{q}+\mathbf{k}}$. In fact, if we compare (32) with Eq.(27), we find that

$$[\hat{\rho}_{\mathbf{k}}, \hat{\rho}_{\mathbf{q}}] = 2ie^{l^2\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\mathbf{k}\wedge\mathbf{q}/2\right)\hat{\rho}_{\mathbf{q}+\mathbf{k}}, \tag{33}$$

which agrees with the result obtained from the LLL projection formalism. In the LLL projection approach, it is proved that the projected electron density operators with different momenta obey an algebra similar to the one of the translation operators in a magnetic field. When a particle in a magnetic field is translated along the parallelogram generated by the vectors $\mathbf{k}l^2$ and $\mathbf{q}l^2$, the particle acquires a phase equal to $\mathbf{q}\wedge\mathbf{k}$. As a consequence of

that, the commutator $[\hat{\rho_{\mathbf{k}}}, \hat{\rho_{\mathbf{q}}}]$ is not equal to zero, contrary to the behavior of the non-projected operators[8].

In the same way, from expressions (27) and (28), we find that the commutator between $\hat{\rho}_{\mathbf{k}}$ and $S^z_{\mathbf{q}}$ is also different from zero,

$$
\begin{aligned}
[\hat{\rho_{\mathbf{k}}}, S^z_{\mathbf{q}}] &= 2ie^{l^2\mathbf{k}\cdot\mathbf{q}/2}\sin(\mathbf{k}\wedge\mathbf{q}/2)e^{-|lk+lq|^2/4}\sum_{\mathbf{p}}\cos\left(\frac{(\mathbf{k}+\mathbf{q})\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}} \\
&= \frac{1}{2}\delta_{\mathbf{k},0}\delta_{\mathbf{q},0}N_\phi^2 - 2ie^{l^2\mathbf{k}\cdot\mathbf{q}/2}\sin(\mathbf{k}\wedge\mathbf{q}/2)\,S^z_{\mathbf{k}+\mathbf{q}}. \qquad (34)
\end{aligned}
$$

This result implies that, within the LLL subspace, the charge and spin excitations are entangled[8]. As it will be discussed in Secs. III C and IV, the charged excitation of the interacting two-dimensional electron gas at $\nu = 1$ is described by a charge spin texture (quantum Hall skyrmion).[16]

Finally, after some algebra, it is possible to show that the commutation relations between the spin operators $S^z_{\mathbf{q}}$, $S^+_{\mathbf{k}}$ and $S^-_{\mathbf{q}}$ are given by

$$
\begin{aligned}
[S^-_{\mathbf{k}}, S^z_{\mathbf{q}}] &= e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\cos(\mathbf{k}\wedge\mathbf{q}/2)\sqrt{N_\phi}e^{-|lk+lq|^2/4}b^\dagger_{\mathbf{k}+\mathbf{q}} \\
&= e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\cos(\mathbf{k}\wedge\mathbf{q}/2)S^-_{\mathbf{k}+\mathbf{q}} \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
[S^+_{\mathbf{k}}, S^z_{\mathbf{q}}] &= -e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\cos(\mathbf{k}\wedge\mathbf{q}/2) \\
&\quad\times\left[\sqrt{N_\phi}e^{-|lk+lq|^2/4}b_{-\mathbf{k}-\mathbf{q}} - \frac{e^{-|lk+lq|^2/4}}{\sqrt{N_\phi}}\sum_{\mathbf{p},\mathbf{p}'}\cos\left(\frac{(\mathbf{k}+\mathbf{q}+\mathbf{p})\wedge(\mathbf{p}-\mathbf{p}')}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{p}'}b_{\mathbf{p}}b_{\mathbf{p}'}\right] \\
&= -e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\cos(\mathbf{k}\wedge\mathbf{q}/2)S^+_{\mathbf{k}+\mathbf{q}} \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
[S^+_{\mathbf{k}}, S^-_{\mathbf{q}}] &= N_\phi e^{-|lk|^2/2}\delta_{\mathbf{q},-\mathbf{k}} - 2e^{-|lk|^2/4-|lq|^2/4}\cos(\mathbf{k}\wedge\mathbf{q}/2)\sum_{\mathbf{p}}\cos\left(\frac{(\mathbf{k}+\mathbf{q})\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}} \\
&\quad+2e^{-|lk|^2/4-|lq|^2/4}\sin(\mathbf{k}\wedge\mathbf{q}/2)\sum_{\mathbf{p}}\sin\left(\frac{(\mathbf{k}+\mathbf{q})\wedge\mathbf{p}}{2}\right)b^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{p}}b_{\mathbf{p}} \\
&= 2e^{l^2\mathbf{k}\cdot\mathbf{q}/2}\cos(\mathbf{k}\wedge\mathbf{q}/2)S^z_{\mathbf{k}+\mathbf{q}} + ie^{l^2\mathbf{k}\cdot\mathbf{q}/2}\sin(\mathbf{k}\wedge\mathbf{q}/2)\hat{\rho}_{\mathbf{k}+\mathbf{q}}. \qquad (37)
\end{aligned}
$$

Again, all the commutation relations (35) - (37) are in agreement with the results calculated in the LLL projection formalism.

It is not surprising that our bosonization approach for the 2DEG at $\nu = 1$ recovers the results obtained with the LLL projection. Remember that all operators considered until this moment were expanded in terms of the fermionic creation and annihilation operators $c^\dagger_{m\,\sigma}$ and $c_{m\,\sigma}$ with the aid of expressions (6), which are the fermionic field operators projected in the LLL. In addition, as discussed in details in the appendix C, the function $G_{m,m'}(x)$ is the matrix element in the lowest Landau level basis of the projected operator $e^{-i\mathbf{q}\cdot\mathbf{r}}$. When the Fourier transform of any operator is calculated using the LLL projection formalism, it is necessary to consider the expression of the projected operator $e^{-i\mathbf{q}\cdot\mathbf{r}}$.

Returning to the previous section, we can also conclude that the operator $S^-_{\mathbf{q}}$ is very well defined by Eq.(29) as this one preserves the commutation relation between the electron density and spin density operators within the LLL subspace.

E. Hilbert space

In the bosonization approach for the one-dimensional electron gas, Haldane[26] proved that the Hilbert space spanned by an arbitrary combination of the fermionic creation and annihilation operators acting on the vacuum state $|N = 0\rangle_0$ is identical to the Hilbert space spanned

by an arbitrary combination of the bosonic creation operators acting on the set of all N-particle ground states $|N\rangle_0$, with $N \in Z$, which is the vacuum state for the bosons.[1]

The above assumption can be elegantly proved by calculating the grand canonical partition functions of the noninteracting fermionic and bosonic Hamiltonians, where the latter is derived from the former using the bosonization method for the one-dimensional electron gas. Since all terms of the partition function are positive quantities, the relation between the two functions allows us to compare the degree of degeneracy of the fermionic and bosonic Hilbert spaces. For the 2DEG at $\nu = 1$, we have been considering a system constituted by a fixed number of particles $N = N_\phi$, therefore we will analyze the canonical partition function.

In Sec.II A, we showed that the Hamiltonian of the 2DEG at $\nu = 1$ is given only by the Zeeman term [see Eq. (9)], which is diagonal in the Landau level basis. The energy eigenvalues can be written as $E_n = ng - gN_\phi/2$, where $0 \leq n \leq N_\phi$ is the number of electrons with spin down. The degeneracy $\mathcal{P}_n^F$ of each energy eigenstate can be easily calculated,

$$\mathcal{P}_n^F = \binom{N_\phi}{n} \cdot \binom{N_\phi}{n} = \left(\frac{N_\phi!}{n!(N_\phi - n)!} \right)^2 .$$

Hence, the fermionic partition function is given by

$$\mathcal{Z}_0^F \;=\; Tr(e^{-\beta \mathcal{H}_0^F}) = e^{\beta g N_\phi/2} \sum_{n=0}^{N_\phi} \binom{N_\phi}{n}^2 e^{-n\beta g}, \quad (38)$$

with $\beta = 1/(K_B T)$.

On the other hand, as it will be discussed in Sec.III A, the bosonic Hamiltonian [Eq. (44)] derived from the noninteracting fermionic one [Eq. (9)] using the bosonization scheme is diagonal in the basis of the eigenstates $|\{n_\mathbf{q}\}\rangle$ [Eq. (21)]. Therefore, the canonical partition function is simply given by

$$\begin{aligned} \mathcal{Z}_0^B &= Tr(e^{-\beta \mathcal{H}_0^B}) \sum_{\{n_\mathbf{q}\}} \langle \{n_\mathbf{q}\}| e^{-\beta \mathcal{H}_0^B} |\{n_\mathbf{q}\}\rangle \\ &= e^{\beta g N_\phi/2} \sum_{n=0}^{N_\phi} \mathcal{P}_n^B e^{-n\beta g}. \end{aligned} \quad (39)$$

Here $\mathcal{P}_n^B$ is the number of eigenstates with n-bosons and $n = \sum_\mathbf{q} n_\mathbf{q}$. Notice that the eigenstates with $n > N_\phi$ are not included in the above sum as those states correspond to a number of electron-hole excitations greater than the number of fermions N_ϕ.

The values of $\mathcal{P}_n^B$ are determined by the number of points in the momentum space. The maximum momentum value can be estimated if we remember that a boson of momentum $\mathbf{q}$, created by the action of the operator $b_\mathbf{q}^\dagger$ on the state $|FM\rangle$, can be described as an electron-hole pair whose distance between the center of their guiding centers is $|\mathbf{r}| = l^2|\mathbf{q}|$. Moreover, as discussed in Appendix B, a particle in the lowest Landau level with guiding center m corresponds to a particle moving in a cyclotron orbit with radius equal to the magnetic length l whose guiding center is located at a distance $l\sqrt{2m+1}$ from the origin of the coordinate system. Therefore the largest distance between the electron and the hole in the magnetic exciton corresponds to $m = N_\phi$ and it is roughly equal to $\sqrt{2N_\phi} l$. Since the momentum cutoff is $q_{max} = \sqrt{2N_\phi}/l$, the number of points in the momentum space is given by

$$\sum_\mathbf{q} 1 = \frac{\mathcal{A}}{4\pi^2} \int d^2q = \frac{2\pi l^2 N_\phi}{4\pi^2} \int_0^{q_{max}} q dq \int d\theta = N_\phi^2, \quad (40)$$

where $\mathcal{A}$ is the system area.

From the above analysis, the number of states with n bosons is given by

$$\begin{aligned} \mathcal{P}_0^B &= 1 \\ \mathcal{P}_1^B &= N_\phi^2 \\ \mathcal{P}_2^B &= N_\phi^2 + \frac{N_\phi^2!}{(N_\phi^2 - 2)!2!} \\ &\dots \\ \mathcal{P}_n^B &= \frac{1}{n!} N_\phi^2 (N_\phi^2 + 1) \dots (N_\phi^2 + (n-1)), \qquad n \geq 1, \end{aligned}$$

hence the canonical partition function for the bosonic Hamiltonian can be written as

$$\mathcal{Z}_0^B = e^{\beta g N_\phi/2} (1 + \sum_{n=1}^{N_\phi} \mathcal{P}_n^B e^{-n\beta g}). \quad (41)$$

Since $\mathcal{Z}_0^B \gg \mathcal{Z}_0^F$, we can conclude that the bosonic Hilbert space is bigger than the femionic one. Even having removed the states with $n \geq N_\phi$ from the partition function (41), we still have unphysical states in the bosonic Hilbert space.

There is only one fermionic and one bosonic subspaces of the corresponding Hilbert spaces which are identical. Notice that the first two terms of equations (38) and (41) are equal, which implies that the fermionic subspace spanned by the quantum Hall ferromagnet, $|FM\rangle$, and the states with only one spin down ($n = 1$) is identical to the bosonic Hilbert space spanned by the vacuum and the states of one-boson $b_\mathbf{q}^\dagger |FM\rangle$.

This overcompleteness of the bosonic Hilbert space can be easily understood. From expressions (38) and (41), we can see that the number of states of two-bosons is roughly twice the number of fermionic states with two spin down ($n = 2$). If we consider, for instance, a state of two-bosons constituted by two bosons of momenta $\mathbf{q_1}$ and $\mathbf{q_2}$, such that $|lq_1|,\ |lq_2| < 1$, in the fermionic language, it can be seen as in Fig. 2.a. Notice that this state is equivalent to the one, which is constituted by two bosons of momenta $\mathbf{q_3}$ and $\mathbf{q_4}$, such that $|lq_3|,\ |lq_4| \gg 1$ [Fig. 2.b]. Based on that, we can say that we are "double counting"

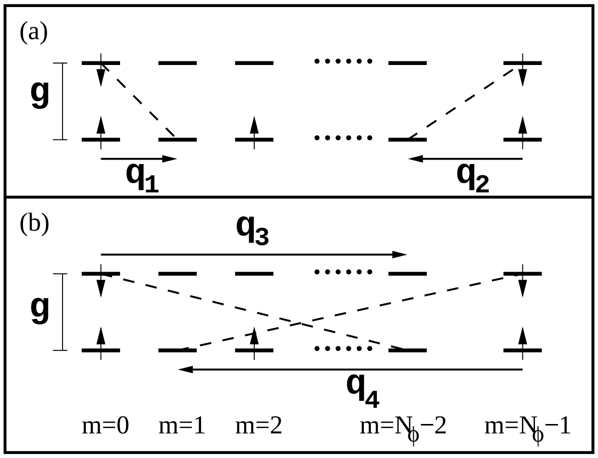

FIG. 2: Schematic representation of a two-bosons state: (a) $|lq_1|,\ |lq_2| < 1$ and (b) $|lq_3|,\ |lq_4| \gg 1$.

the number of states of two-bosons. Of course, this problem becomes worse as we take into account states of n bosons ($n > 2$). Besides that, we can still study the low-energy physics of the system as, in this case, we have a small number of bosons with momentum $|lq| < 1$ present in the system. As we will see in the next section, the energy of the bosons increases with the momentum [Eq. (51)]. It is worth mentioning that this same problem appears in a description of a bilayer quantum Hall system at total filling factor one ($\nu_T = 1$).[27] Here, the sponaneous interlayer phase coherent (111) state can be viewed as a condensate of interlayer particle-hole pairs (excitons), which, in the very dilute regime, can be treated as point-like bosons. The corresponding bosonic Hilbert space is also overcomplete.

This problem could be fixed, for instance, introducing a constraint which restricts the bosonic Hilbert space to the physical states only. However, this is quite a hard task. For example, it is *not* possible to follow the ideas of the well-known expansions of the spin operators in terms of bosons, such as the Schwinger boson representation.[28] In this case, the *local* spin operators are written as a function of the *local* bosons operators a_i and $\tilde{a}_i$, namely

$$S_i^+ = a_i^\dagger \tilde{a}_i, \qquad S_i^- = \tilde{a}_i^\dagger a_i, \qquad S_i^z = (a_i^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_i)/2.$$

The constraint is easily determined as it is related to the fact that the number of bosons on the site i should be twice the spin S, i.e., $a_i^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_i = 2S$. The same idea *can not* be applied to our case as the bosonic operators $b_{\mathbf{q}}^\dagger$ and $b_{\mathbf{q}}$ are not local. In fact, they involve a linear combination of electron-hole excitations where the particles are localized in different guiding centers. Until now, we have not found a systematic way of introducing a constraint in our formalism.

III. INTERACTING TWO-DIMENSIONAL ELECTRON GAS AT $\nu = 1$

In this section, we will apply the bosonization method developed for the 2DEG at $\nu = 1$ to study the interacting two-dimensional electron gas at $\nu = 1$. We will show that the Hamiltonian of this interacting system is mapped into an interacting bosonic model.

A. Noninteracting electron system

As pointed out in the last section, the Hamiltonian of the noninteracting two-dimensional electrons at $\nu = 1$, restricted to the lowest Landau level subspace, is given by the Zeeman term only [Eq. (9)]. In the Landau level basis, it can be written as

$$\mathcal{H}_0 \equiv \mathcal{H}_Z = -\frac{1}{2} g \sum_\sigma \sum_m \sigma c_{m\,\sigma}^\dagger c_{m\,\sigma}, \tag{42}$$

where $g = g^* \mu_B B > 0$.

In order to find out the bosonic form of the Hamiltonian (42), it is necessary to calculate the commutation relation between $\mathcal{H}_0$ and the bosonic creation operator $b_{\mathbf{q}}^\dagger$,

$$\begin{aligned}
[\mathcal{H}_0, b_{\mathbf{q}}^\dagger] &= -\frac{1}{2} g \sum_\sigma \sum_{m,n,n'} \frac{e^{-|lq|^2/4}}{\sqrt{N_\phi}} \sigma G_{n,n'}(lq) \\
&\quad \times [c_{m\,\sigma}^\dagger c_{m'\,\sigma}, c_{n\,\downarrow}^\dagger c_{n'\,\uparrow}] \\
&= g \sum_{n,n'} \frac{1}{\sqrt{N_\phi}} e^{-|lq|^2/4} G_{n,n'}(lq) c_{n\,\downarrow}^\dagger c_{n'\,\uparrow} \\
&= g b_{\mathbf{q}}^\dagger.
\end{aligned} \tag{43}$$

Since the above commutator is proportional to $b_{\mathbf{q}}^\dagger$, $\mathcal{H}_0$ should present a term of the form $g \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}}$, which gives the same commutation relation as in Eq. (43). Moreover, the action of $\mathcal{H}_0$ on $|FM\rangle$ is equal to a constant, $-gN_\phi/2$. Therefore we can conclude that the bosonic form of the Zeeman term is

$$\mathcal{H}_0 = g \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} - \frac{1}{2} g N_\phi. \tag{44}$$

The same result can be obtained in a more rigorous way by explicitly calculating the action of $\mathcal{H}_0$ on the eigen-

states (21),

$$\begin{aligned}
\mathcal{H}_0|\{n_{\mathbf{q}}\}\rangle &= \mathcal{H}_0(\prod_{\mathbf{q}} \frac{(b^\dagger_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}|FM\rangle) \qquad (45)\\
&= [\mathcal{H}_0, \prod_{\mathbf{q}} \frac{(b^\dagger_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}]|FM\rangle + \prod_{\mathbf{q}} \frac{(b^\dagger_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}\mathcal{H}_0|FM\rangle\\
&= (g\sum_{k\in\{n_{\mathbf{q}}\}} n_{\mathbf{k}} - \frac{1}{2}gN_\phi)\prod_{\mathbf{q}} \frac{(b^\dagger_{\mathbf{q}})^{n_{\mathbf{q}}}}{\sqrt{n_{\mathbf{q}}!}}|FM\rangle.
\end{aligned}$$

This analysis shows that the Hamiltonian of the noninteracting two-dimensional electron gas at $\nu = 1$, restrict to the lowest Landau level is recast in a noninteracting bosonic system, whose dispersion relation is constant.

B. Interacting electron system

Now, we will consider an interacting two-dimensional electron gas at $\nu = 1$, where the particles are restricted to the lowest Landau level. The Hamiltonian of the system is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}. \qquad (46)$$

Here, $\mathcal{H}_0$ is given by Eq. (42) and the interacting term can be written as

$$\mathcal{H}_{int} = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^2r d^2r'\ \Psi^\dagger_\sigma(\mathbf{r})\Psi^\dagger_{\sigma'}(\mathbf{r'})V(|\mathbf{r}-\mathbf{r'}|)\Psi_{\sigma'}(\mathbf{r'})\Psi_\sigma(\mathbf{r}), \qquad (47)$$

where $V(|\mathbf{r}|) = e^2/(\epsilon r)$ is the Coulomb potential and ϵ the dielectric constant of the host semiconductor (see Appendix A). Substituting Eq.(6) in $\mathcal{H}_{int}$, it is possible to write down the interacting term as a function of the density operators of electrons σ as

$$\mathcal{H}_{int} = \frac{1}{2}\sum_{\sigma,\sigma'}\int \frac{d^2k}{4\pi^2}\, V(k)\rho_\sigma(\mathbf{k})\rho_{\sigma'}(-\mathbf{k}) \qquad (48)$$

where $V(k)$ is the Fourier transform of the Coulomb potential in 2D,

$$V(k) = \frac{2\pi e^2}{\epsilon k},$$

and $k = |\mathbf{k}|$. Using the bosonic form of the density operators $\hat{\rho}_\sigma(\mathbf{k})$, we can write down the interacting term as a function of the bosonic creation and annihilation operators. Substituting Eqs. (25) and (26) in Eq. (48), we have four distinct terms, $\mathcal{H}_{int} = \mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4$. The first one is a constant related to the positive background,

$$\mathcal{H}_1 = \frac{1}{8\pi^2}\int d^2k\, V(\mathbf{k})\delta_{\mathbf{k},0},$$

whereas the second and third terms are equal to zero as

$$\begin{aligned}
\mathcal{H}_2 = -\mathcal{H}_3 &= -\frac{i}{4\pi^2}\sum_{\mathbf{p}}\int d^2k\, V(\mathbf{k})N_\phi\delta_{\mathbf{k},0}e^{-|lk|^2/4}\\
&\quad\times\sin(\mathbf{k}\wedge\mathbf{p}/2)b^\dagger_{\mathbf{q}}b_{\mathbf{q}}.
\end{aligned}$$

The last term is *quartic* in the bosonic operators. Rewriting $\mathcal{H}_4$ in normal-ordering in the operators b, we end up with a quadratic and a quartic terms in the bosonic operators, namely

$$\begin{aligned}
\mathcal{H}_4 &= \frac{1}{2\pi^2}\sum_{\mathbf{p},\mathbf{q}}\int d^2k\, V(\mathbf{k})e^{-|lk|^2/2}\sin(\mathbf{k}\wedge\mathbf{p}/2)\\
&\quad\times\sin(\mathbf{k}\wedge\mathbf{q}/2)b^\dagger_{\mathbf{k+q}}b_{\mathbf{q}}b^\dagger_{\mathbf{p-k}}b_{\mathbf{p}}\\
&= \frac{1}{2\pi^2}\sum_{\mathbf{q}}\int d^2k\, V(\mathbf{k})e^{-|lk|^2/2}\sin^2(\mathbf{k}\wedge\mathbf{q}/2)b^\dagger_{\mathbf{q}}b_{\mathbf{q}}\\
&\quad+\frac{1}{2\pi^2}\sum_{\mathbf{p},\mathbf{q}}\int d^2k\, V(\mathbf{k})e^{-|lk|^2/2}\sin(\mathbf{k}\wedge\mathbf{p}/2)\\
&\quad\times\sin(\mathbf{k}\wedge\mathbf{q}/2)b^\dagger_{\mathbf{k+q}}b^\dagger_{\mathbf{p-k}}b_{\mathbf{q}}b_{\mathbf{p}}\\
&= \frac{e^2}{\epsilon l}\sqrt{\frac{\pi}{2}}\sum_{\mathbf{q}}\left(1-e^{-|lq|^2/4}I_0(|lq|^2/4)\right)b^\dagger_{\mathbf{q}}b_{\mathbf{q}}\\
&\quad+\frac{1}{2\pi^2}\sum_{\mathbf{p},\mathbf{q}}\int d^2k\, V(\mathbf{k})e^{-|lk|^2/2}\sin(\mathbf{k}\wedge\mathbf{p}/2)\\
&\quad\times\sin(\mathbf{k}\wedge\mathbf{q}/2)b^\dagger_{\mathbf{k+q}}b^\dagger_{\mathbf{p-k}}b_{\mathbf{q}}b_{\mathbf{p}}. \qquad (49)
\end{aligned}$$

Here $I_0(x)$ is the modified Bessel function of the first kind.[29]

Now, adding the bosonic form of the noninteracting Hamiltonian [Eq. (44)] to $\mathcal{H}_{int}$, we will arrive at the expression of the total Hamiltonian of the interacting elec-

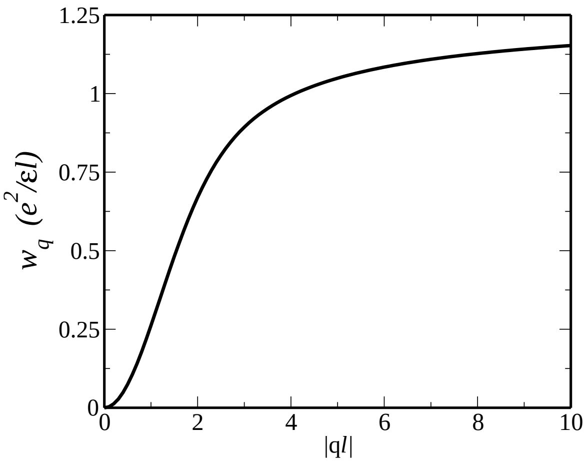

FIG. 3: Dispersion relation of the bosons [Eq. (51), with $g = 0$], in units of the Coulomb energy $e^2/(\epsilon l)$ as a function of the momentum $\mathbf{q}$.

trons as a function of the bosonic operators,

$$\begin{aligned}\mathcal{H} &= -\frac{1}{2}gN_\phi + \sum_{\mathbf{q}} w_{\mathbf{q}} b^\dagger_{\mathbf{q}} b_{\mathbf{q}} \\ &\quad + \frac{2}{\mathcal{A}} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} V(k) e^{-|lk|^2/2} \sin(\mathbf{k}\wedge\mathbf{p}/2) \\ &\quad \times \sin(\mathbf{k}\wedge\mathbf{q}/2) b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{\mathbf{p}-\mathbf{k}} b_{\mathbf{q}} b_{\mathbf{p}},\end{aligned} \tag{50}$$

where the dispersion relation of the bosons is given by

$$w_{\mathbf{q}} = g + \frac{e^2}{\epsilon l}\sqrt{\frac{\pi}{2}}\left(1 - e^{-|lq|^2/4} I_0(|lq|^2/4)\right). \tag{51}$$

This curve is plotted in Fig. 3 for the case $g = 0$. The energy of the bosons is given in units of the Coulomb energy $e^2/(\epsilon l)$.

The Hamiltonian (50) describes a system of interacting two-dimensional bosons. The ground state is the vacuum state $|FM\rangle$ [Eq. (10)] and its energy is equal to $E_0 = -\frac{1}{2}gN_\phi$. This result implies that the ground state of the two-dimensional electron gas at $\nu = 1$ does not change as we moved from the noninteracting to the interacting system. As pointed out earlier, the ground state of the interacting two-dimensional electron gas is the quantum Hall ferromagnet even in the limit of vanishing Zeeman energy ($g \to 0$).

The elementary neutral excitations are described by the bosons b, whose dispersion relation $w_{\mathbf{q}}$ is equal to the previous diagrammatic calculations of Kallin and Halperin[13] and the results of Bychkov *et al.*[14] The long wavelength excitations can be considered as the spin wave excitations of the quantum Hall ferromagnet, while the ones with large momenta correspond to a quasielectron-quasihole pair as the particles are very far apart. Remember that, as discussed in Sec. II A, the distance between the guiding centers of the excited electron and hole is $|\mathbf{r}| = l^2|\mathbf{q}|$.

The above results corroborate our initial association [Sec. II A] between the bosons b and the magnetic excitons described in Ref.13. As a matter of fact, within our bosonization method, we goes beyond the diagrammatic calculation[13] as we end up with an interaction between the bosons. In the next section, we will study the resulting interacting bosonic model.

C. Bound states of two-bosons

In this section, we will study the states of two-bosons. More precisely, we will check if the interacting bosonic model (50) can describe the formation of bound states of two-bosons. Our initial motivation is based on previous results of the one-dimensional ferromagnetic Heisenberg model.

The problem of two interacting spin waves (magnons) in the ferromagnetic Heisenberg model was analyzed by Dyson[23], who derived a bound state condition when the total momentum of the pair is equal to zero. For an arbitrary value of the spin S, it was verified that this condition is not fulfilled in two and three dimensions and it was concluded that bound states of two spin waves do not exist, in contradiction to the results for the one-dimensional model.[30] After that, this problem was also discussed by Wortis[31] who, in opposition to Dyson's results, proved the existence of bound states of two spin waves for any value of the spin S and the dimensionality of the system. A review about this topic is presented in Ref. 24.

On the other hand, Tjon and Wright[32] studied the dynamical solitons of the one-dimensional ferromagnetic Heisenberg model. The dynamical soliton is a solution of the dynamical equations of motion, localized in space, with zero topological charge and whose total energy, total field momentum and z-component of the total magnetization,

$$M^z = \int dx\; S^z(x),$$

are constants of motion. Writing the components of the spin operator as $S^x + iS^y = Se^{i\phi}\sin\theta$ and $S^z = S\cos\theta$, where S is the spin of the system, the general form of the dynamical soliton can be written as[33]

$$\theta = \theta(\mathbf{r} - \mathbf{v}t), \quad \phi = wt + \phi(\mathbf{r} - \mathbf{v}t).$$

Here, $\mathbf{v}$ is the translational velocity of the soliton and w is the precessional frequency of the magnetization in the frame of reference moving with the soliton, i.e., an internal degree of freedom.

A possible relation between the dynamical soliton and the bound states of n-magnons of the Heisenberg model was discussed by Schneider.[34] For the isotropic model, the dynamical soliton solutions of Tjon and Wright[32] were semiclassically quantized via the Bohr-Sommerfeld-de Broglie condition. Following this procedure, the z-

component of the total magnetization assumes only integer values. Consequently, the precessional frequency w is also quantized.[34] For spin $S = 1/2$, it was found a correspondence between the n-magnons and the dynamical soliton spectra. This result implies that the dynamical soliton of the one-dimensional ferromagnetic Heisenberg model can be considered as a bound state of n-magnons. This analysis was also applied to the easy-axis and anisotropic exchange Heisenberg models, but the correspondence between the two spectra was found only in the limit of large quantum numbers.

Those characteristics are similar to the solutions of the Sine-Gordon model.[35] Two possible solutions of the classical equations of motion are the topological soliton (a static localized solution) and the breather solution, which resembles an oscillating soliton-antisoliton pair (dynamical soliton) with topological charge equal to zero (a good review about topological and dynamical solitons is presented in Ref. 33). After the quantization of these solutions, the soliton corresponds to a quasiparticle (fermion of the massive Thirring model) while the internal degrees of freedom (oscillating modes) of the breather solution correspond to bound states. Since the lowest energy bound state can be considered as an *elementary boson* of the theory, the internal degrees of freedom of the breather solution correspond to bound states of n-bosons. The number of the latter is determined by the coupling constant of the theory.

It is well known that the low lying charged excitations of the two-dimensional electron gas at $\nu = 1$ is the quantum Hall skyrmion, which carries an unusual spin distribution.[16] As we will see in the next section, this excitation is described by a generalized nonlinear sigma model in terms of a unit vector field $\mathbf{n}(\mathbf{r})$ which is related to the electronic spin [see Eq. (62)]. The skyrmion is given by the topological soliton solution of the nonlinear sigma model with a finite size, which is determined by the competition between the Coulomb and Zeeman energies. For $\nu = 1$, the topological charge of this solution is equal to the electrical charge.

As our bosonization method for the 2DEG at $\nu = 1$ gives us an interacting boson model to describe the interacting two-dimensional electron gas at $\nu = 1$, it seems reasonable to study the bound states of two-bosons in order to find out a possible relation between them and the spectrum of a bound skyrmion-antiskyrmion pair.

Since the total and relative momenta of a boson pair are given by $\mathbf{P} = \mathbf{p} + \mathbf{q}$ and $\mathbf{Q} = (\mathbf{p} - \mathbf{q})/2$ respectively, the interacting term of the bosonic Hamiltonian (50) can be written as

$$\begin{aligned}\mathcal{H}_{int} &= \frac{2}{\mathcal{A}} \sum_{\mathbf{k},\mathbf{P},\mathbf{Q}} V(k) e^{-|lk|^2/2} \sin\left(\frac{\mathbf{k} \wedge (\mathbf{P/2} + \mathbf{Q})}{2}\right) \\ &\times \sin\left(\frac{\mathbf{k} \wedge (\mathbf{P/2} - \mathbf{Q})}{2}\right) \\ &\times b^\dagger_{\mathbf{P/2+k-Q}} b^\dagger_{\mathbf{P/2-k+Q}} b_{\mathbf{P/2+Q}} b_{\mathbf{P/2-Q}}. \end{aligned} \tag{52}$$

We can easily see that a state of two-bosons of the kind $|\Phi\rangle = b^\dagger_{\mathbf{q}} b^\dagger_{\mathbf{p}} |FM\rangle$ is not an eigenstate of the total Hamiltonian (50). Therefore, it is necessary to consider a linear combination of those states. The more general form of a state of two-bosons with total momentum $\mathbf{P}$ can be written as

$$|\Phi_{\mathbf{P}}\rangle = \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}} |FM\rangle. \tag{53}$$

In this case, the total momentum of the pair is also a good quantum number for the same reasons as discussed at the end of Sec. II A. Remember that a state of two-bosons can be considered as a two electron-hole pairs whose total charge is zero.

For a fixed value of the total momentum $\mathbf{P}$, the energy of the state (53) is given by the Schrödinger equation

$$\mathcal{H}|\Phi_{\mathbf{P}}\rangle = E_{\mathbf{P}}|\Phi_{\mathbf{P}}\rangle. \tag{54}$$

We will consider the action on $|\Phi_{\mathbf{P}}\rangle$ of the quadratic and quartic terms of the total Hamiltonian (50) separately. For $\mathcal{H}_0$, we have

$$\begin{aligned}\mathcal{H}_0|\Phi_{\mathbf{P}}\rangle &= \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) [\mathcal{H}_0, b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}}] |FM\rangle \\ &+ \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}} \mathcal{H}_0 |FM\rangle \\ &= \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) \underbrace{(w_{\frac{1}{2}\mathbf{P}-\mathbf{q}} + w_{\frac{1}{2}\mathbf{P}+\mathbf{q}})}_{E_{\mathbf{P}}(\mathbf{q})} \\ &\times b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}} |FM\rangle \\ &+ \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}} E_{FM} |FM\rangle \\ &= (E_{\mathbf{P}}(\mathbf{q}) + E_{FM})|\Phi_{\mathbf{P}}\rangle. \end{aligned} \tag{55}$$

Observe that $E_{\mathbf{P}}(\mathbf{q})$ is the energy of two noninteracting bosons. On the other hand, for the $\mathcal{H}_{int}$, after some algebra, it is possible to show that

$$\begin{aligned}\mathcal{H}_{int}|\Phi_{\mathbf{P}}\rangle &= \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) [\mathcal{H}_{int}, b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}}] |FM\rangle \\ &+ \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q}) b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}} \mathcal{H}_{int} |FM\rangle \\ &= 2 \sum_{\mathbf{k}\neq\mathbf{0},\mathbf{q}} U(\mathbf{k},\mathbf{P},\mathbf{q}) \Phi_{\mathbf{P}}(\mathbf{q}) \\ &\times b^\dagger_{\frac{1}{2}\mathbf{P}-\mathbf{q}+\mathbf{k}} b^\dagger_{\frac{1}{2}\mathbf{P}+\mathbf{q}-\mathbf{k}} |FM\rangle, \end{aligned} \tag{56}$$

where

$$\begin{aligned}U(\mathbf{k},\mathbf{P},\mathbf{q}) &= \frac{2}{\mathcal{A}} V(k) e^{-|lk|^2/2} \sin\left(\frac{\mathbf{k} \wedge (\mathbf{P/2} + \mathbf{q})}{2}\right) \\ &\times \sin\left(\frac{\mathbf{k} \wedge (\mathbf{P/2} - \mathbf{q})}{2}\right). \end{aligned}$$

Substituting expressions (55) and (56) in the Schrödinger equation (54) and changing $\mathbf{q} \rightarrow \mathbf{q} + \mathbf{k}$, we have

$$\begin{aligned} 0 &= \sum_{\mathbf{q}} \Phi_{\mathbf{P}}(\mathbf{q})(E_{\mathbf{P}}(\mathbf{q}) + E_{FM} - E_{\mathbf{P}}) b^{\dagger}_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^{\dagger}_{\frac{1}{2}\mathbf{P}+\mathbf{q}} |FM\rangle \\ &+ 2 \sum_{\mathbf{k}\neq \mathbf{0},\mathbf{q}} U(\mathbf{k},\mathbf{P},\mathbf{q}) \Phi_{\mathbf{P}}(\mathbf{q}+\mathbf{k}) b^{\dagger}_{\frac{1}{2}\mathbf{P}-\mathbf{q}} b^{\dagger}_{\frac{1}{2}\mathbf{P}+\mathbf{q}} |FM\rangle. \end{aligned}$$

Changing the sum over momenta to an integral

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} \rightarrow \int \frac{d^2 q}{4\pi^2},$$

we find the following eigenvalue problem

$$(\epsilon - E_{\mathbf{P}}(\mathbf{q}))\Phi_{\mathbf{P}}(\mathbf{q}) = \int d^2k\, K_{\mathbf{P}}(\mathbf{k}-\mathbf{q},\mathbf{q})\Phi_{\mathbf{P}}(\mathbf{k}), \quad (57)$$

where $\epsilon = E_{\mathbf{P}} - E_{FM}$ and the kernel of the integral equation is given by

$$\begin{aligned} K_{\mathbf{P}}(\mathbf{k}-\mathbf{q},\mathbf{q}) &= 2\frac{\epsilon_c}{\pi} \frac{e^{-|\mathbf{k}-\mathbf{q}|^2/2}}{|\mathbf{k}-\mathbf{q}|} \\ &\times \sin\left(\frac{(\mathbf{k}-\mathbf{q}) \wedge (\mathbf{P/2}+\mathbf{q})}{2}\right) \\ &\times \sin\left(\frac{(\mathbf{k}-\mathbf{q}) \wedge (\mathbf{P/2}-\mathbf{q})}{2}\right). \quad (58) \end{aligned}$$

In the two expressions above, all momenta are measured in units of the inverse magnetic length, i.e., $\mathbf{q} \rightarrow \mathbf{q}/l$.

For the one-dimensional Heisenberg model, the analog eigenvalue problem can be solved analytically as the kernel of the integral equation is separable.[24,31] However, $K_{\mathbf{P}}(\mathbf{k}-\mathbf{q},\mathbf{q})$ is not of the same kind and therefore our eigenvalue problem (57) will be solved numerically.

The numerical solution of the above eigenvalue problem can be determined using the quadrature technique.[29] This method consists of replacing the integral over momentum by a set of algebraic equations

$$(\epsilon - E_{\mathbf{P}}(\mathbf{q_i}))\Phi_{\mathbf{P}}(\mathbf{q_i}) \approx \sum_{j\neq i} C_j K_{\mathbf{P}}(\mathbf{q_j}-\mathbf{q_i},\mathbf{q_i})\Phi_{\mathbf{P}}(\mathbf{q_j}), \quad (59)$$

where C_j are the quadrature coefficients. The system of equations can be symmetrized multiplying then by $\sqrt{C_j}$,

$$\begin{aligned} &(\epsilon - E_{\mathbf{P}}(\mathbf{q_i}))(C_i^{1/2}\Phi_{\mathbf{P}}(\mathbf{q_i})) \approx \\ &\sum_{j\neq i} C_i^{1/2} K_{\mathbf{P}}(\mathbf{q_j}-\mathbf{q_i},\mathbf{q_i}) C_j^{1/2} (C_j^{1/2}\Phi_{\mathbf{P}}(\mathbf{q_j})). \quad (60) \end{aligned}$$

After this discretization, for a fixed value of the total momentum $\mathbf{P}$, we can calculate the eigenvalues of the equation (60) using usual matrix techniques.

The choice of the points $\mathbf{q_i}$ and of the values of the coefficients C_i are related to the parametrization adopted. For one-dimensional problems, there are several parametrizations, for instance, the Gaussian quadrature[29] which allows us to calculate the eigenvalues with good precision. However, for two-dimensional problems, there are a fewer number of available parametrizations and therefore it is possible only to find a good estimate for the eigenvalues.

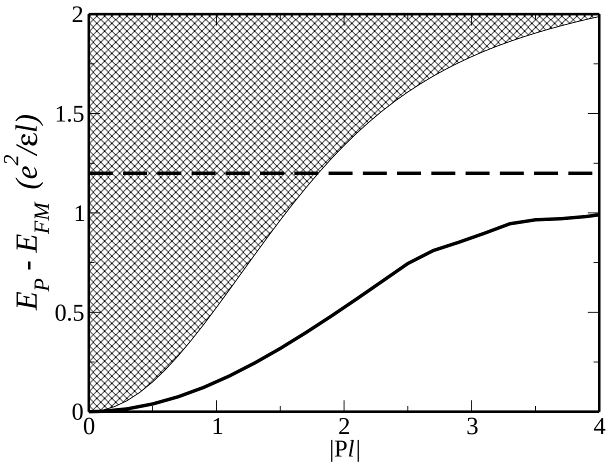

FIG. 4: Dispersion relation of the state of two-bosons, in units of the Coulomb energy $e^2/(\epsilon l)$ as a function of the total momentum $\mathbf{P}$ for $g = 0$. Solid line: lowest energy bound state; shaded area: continuum of the scattering states; dashed line: energy of the quasielectron-quasihole pair plus one spin wave [Eq. (61)]. See text for details.

In order to solve the eigenvalue problem (60) we considered a parametrization which is applied to calculate two-dimensional integrals over a circular region. In this case, it is necessary to introduce a cutoff for large momentum in order to define the integration region. All the parameters of this quadrature as well as a set of parametrizations for multiple integral calculations can be found in Ref. 36.

Fig. 4 shows the dispersion relation of the states of two-bosons as a function of the total momentum $\mathbf{P}$. Here, we assume that $g = 0$. The eigenvalue problem was solved using a 61 point quadrature and only bosons with momentum $|\mathbf{k}l| \leq 2$ were considered. The solid line is the lowest energy eigenvalue state while the shaded area is the continuum of scattering states. Once the lowest energy state of two-bosons is below the continuum of scattering states, we can say that this lowest energy eigenvalue (solid line) corresponds to a bound state of two-bosons. There are also other bound states above the one shown in Fig. 4, but the analysis of those states is limited by the numerical method.

As pointed out at the begining of this section, we want to check if there is a possible relation between the bound states of two-bosons and a bound skyrmion-antiskyrmion pair excitation. In this way, we should compare our results with the ones derived from the model of Sondhi *et al.*[16], namely, with the value of the energy of a noninteracting skyrmion-antiskyrmion pair, which can be cal-

culated from the expression derived by Sondhi *et al.* for the energy of the skyrmion [Eq. (7) of Ref.16]. However, this kind of comparison is not appropriate here as the z component of the total spin of the system is a good quantum number. Notice that the states of two-bosons has $S^z = 2$ (the z component of the total spin in relation to the quantum Hall ferromagnet) while a skyrmion-antiskyrmion pair described by the Sondhi's model has $S^z \gg 2$. Remember that this model is suitable to describe the skyrmion only in the limit of very small Zeeman energy ($g \to 0$). In this case, the excitation is constituted by a large number of spin-flips.

In a work previous to[16], Rezayi[37] constructed a family of wave functions for the quasiparticles of the 2DEG at $\nu = 1$. Based on numerical calculations, it was shown that the energy (E_1) of a state formed by the quantum Hall ferromagnet plus one spin down electron [see Fig. 5(a)] is greater than the energy (E_2) of the state constituted by one spin down electron plus a spin-wave excitation [Fig. 5(b)]. In the thermodynamic limit, it was shown that $E_1 - E_2 = 0.054e^2/\epsilon l$. This result implies that instead of a single spin down quasielectron, the quasiparticle of the 2DEG at $\nu = 1$ should be constituted by a quasielectron bound to n-spin waves. Based on that, Sondhi and coworkers suggested that the charged excitation of the 2DEG at $\nu = 1$ should be described by a charged spin texture.

Notice that we can compare the spectrum of the bound states of two-bosons with Rezayi numerical results. Let us consider a state constituted by a quasielectron-quasihole pair very far apart and a spin wave with momentum $|lk| \ll 1$ bound to either the quasielectron or the quasihole. The z component of the total spin of this state is $S^z = 2$. Since the energy of a quasielectron-quasihole pair very far apart corresponds to the limit $|lk| \to \infty$ of the dispersion relation (51), the energy of the state

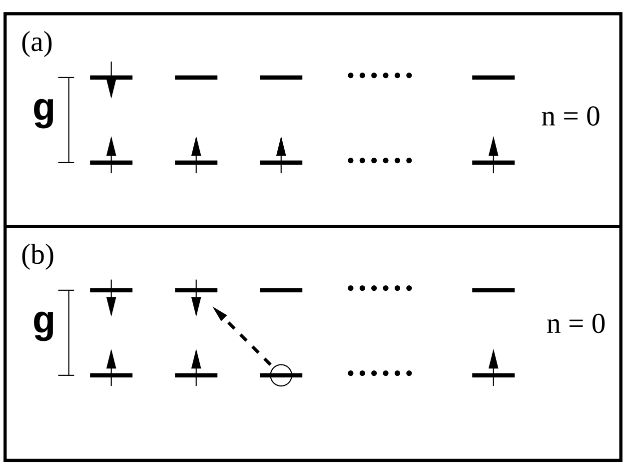

FIG. 5: Schematic representation of the quasiparticules considered by Rezayi in Ref. 37: (a) the quantum Hall ferromagnet plus one spin down electron and (b) the spin down electron plus a spin-wave excitation.

describe above is simply

$$E_{e-h-sw} \approx (\sqrt{\pi/2} - 0.054)e^2/\epsilon l. \tag{61}$$

The dashed line on Fig. 4 corresponds to E_{e-h-sw}. We can see that our results are in good agreement with the previous ones of Rezayi's. More precisely, our calculations indicated that, in the limit $|lP| \to \infty$, the dispersion relation of the bound states of two-bosons is asymptotic to E_{e-h-sw}. In this scenario, we can understand the behavior of the dispersion relation of the bound states of two-bosons. As the total momentum $|lP|$ decreases, for instance, the quasielectron bound to the spin wave approaches the quasihole, increasing the interaction between them and therefore lowering the energy of the system. Notice that this behavior is in good agreement with the solid line on Fig. 4.

Therefore we can conclude that the bound states of two-bosons are appropriate to describe the skyrmion-antiskyrmion pair excitation, in the limit of large Zeeman energy, when the excitation is formed by a small number of spin-flips.

We should mention that Cooper[38] studied the dynamical soliton solutions with zero topological charge of the nonlinear sigma model without the extra terms of the Sondhi's model. The calculated spectrum is qualitative similar to the one illustrated in Fig. 4. For small momentum, the excitations correspond to free spin waves and, as the momentum increases, the dispersion relation continuously approaches the energy value of a noninteracting soliton-antisoliton pair. However, this description is valid only in the limit of large number of spin waves, which is very far from the region of our analysis.

A final word about the Hilbert space. Notice that in the above analysis we consider the bosonic Hilbert space constituted by the vacuum state [Eq. (10)] and the states of two-bosons [Eq.(53)]. As discussed in Sec.II E, the number of the states of two-boson is greater than the number of fermionic states with two spin-flips over the quantum Hall ferromagnet. However, in order to solve the eigenvalue problem (60), a cutoff for large boson momentum was introduced which restricted the bosonic subspace and therefore we believe that the solution of Eq. (60) does not involve unphysical states.

IV. BOSONIZATION AND COHERENT STATES

In this section, we will consider the semiclassical limit of the interacting bosonic Hamiltonian (50). We will show that, starting from (50), it is possible to recover the energy functional of the quantum Hall skyrmion.

As mentioned in Sec. III C, Sondhi *et al.*[16] suggested that the quantum Hall skyrmion can be described by a generalized nonlinear sigma model in terms of an unit vector field $\mathbf{n}(\mathbf{r})$ which is related to the electronic spin. The effective Lagrangean density of the model is given

by

$$\mathcal{L}_{eff} = \frac{1}{2}\rho_0\mathcal{A}(\mathbf{n})\cdot\partial_t\mathbf{n} - \frac{1}{2}\rho_S(\nabla\mathbf{n})^2 + \frac{1}{2}g^*\rho_0\mu_B\mathbf{n}\cdot\mathbf{B} - \frac{e^2}{2\epsilon}\int d^2r'\,\frac{q(\mathbf{r})q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \qquad (62)$$

Here, ρ_S is the *spin stiffness* (see Appendix A), $\mathcal{A}(\mathbf{n})$ is the vector potential of an unit monopole ($\epsilon^{abc}\partial_a\mathcal{A}_b = n^c$), $\rho_0 = 1/(2\pi l^2)$ is the average electronic density, and $q(\mathbf{r})$ is the topological charge density or skyrmion density which is given by

$$q(\mathbf{r}) = \frac{1}{8\pi}\epsilon^{\alpha\beta}\mathbf{n}\cdot(\partial_\alpha\mathbf{n}\times\partial_\beta\mathbf{n}), \qquad (63)$$

with $a,b,c = x,y,z$, $\alpha,\beta = x,y$, and $\epsilon^{\alpha\beta}$ is the antisymmetric tensor.

On the other hand, Moon and co-workers suggested an alternative approach to study the quantum Hall skyrmion.[18] In this case, the charged excitation is described by the state

$$|\mathbf{n}(\mathbf{r})\rangle = e^{-i\mathcal{O}}|FM\rangle, \qquad (64)$$

where the operator $\mathcal{O}$ is a nonuniform spin rotation which reorients the local spin from the direction $\hat{z}$ to $\hat{n}(\mathbf{r})$,

$$\mathcal{O} = \int d^2r\;\mathbf{\Omega}(\mathbf{r})\cdot\mathbf{S}(\mathbf{r}) = \int d^2q\;\left[\Omega^-(\mathbf{q})S^+_{-\mathbf{q}} + \Omega^+(\mathbf{q})S^-_{-\mathbf{q}}\right]. \qquad (65)$$

Here, $\mathbf{S}(\mathbf{r})$ is the spin operator, $\mathbf{n}(\mathbf{r})$ is a unit vector, and $\mathbf{\Omega}(\mathbf{r}) = \hat{z}\times\mathbf{n}(\mathbf{r})$ defines the rotation angle. Assuming that $\mathbf{\Omega}(\mathbf{r})$ corresponds to small tilts away from the $\hat{z}$ direction, $\Omega^\sigma(\mathbf{q})$ vanishes when $|lq| \gg 1$.

In this long wavelength approximation, it was shown that the average value of the electron density operator in the state (64) is equal to the topological charge density of the vector field $\mathbf{n}(\mathbf{r})$ [Eq. (63)]. Moreover, after projecting the Coulomb potential in the lowest Landau level subspace, its average value in the state (64) is equal to functional energy derived from the Lagrangean density (62).

In Sec. III C, we showed a possible relation between the skyrmion-antiskyrmion pair excitation and the bound states of two bosons. Therefore if we consider a semiclassical limit of the interacting bosonic Hamiltonian (50) in the same way as it was done in Ref. 18 we can check if it is possible to recover the results of Sondhi *et al.*[16]

Let us assume that equation (64) is a good approximation to describe the skyrmion. Substituting the expressions (31) and (29) in (65) and approximating $S^+_{\mathbf{q}}$ only by the linear term in the bosonic annihilation operator, we can write down the state (64) as a function of the bosonic operator b as

$$|sk\rangle = e^{-\mathcal{N}}e^{-i\mathcal{O}}|FM\rangle, \qquad (66)$$

where the operator $\mathcal{O}$ is redefined as

$$\mathcal{O} \equiv \frac{1}{8\pi^2}\sqrt{\beta N_\phi}\int d^2q\;\Omega^+_{\mathbf{q}}b^\dagger_{\mathbf{q}}, \qquad (67)$$

and the constant $\mathcal{N} = \frac{\beta N_\phi}{2(8\pi)^2}\int d^2q\;\Omega^+_{\mathbf{q}}\Omega^-_{-\mathbf{q}}$. The value of the constant β will be determined later. Observe that the state (66) is a coherent state of the bosons b.

Changing the sum over momenta into an integral in the expression of the bosonic representation of the electron density operator $\hat{\rho}_{\mathbf{k}}$ [Eq. (27)] the average value of this operator in the state (66) is given by

$$\langle sk|\hat{\rho}_{\mathbf{k}}|sk\rangle = i\frac{1}{2}\frac{N_\phi^2\beta}{(2\pi)^5}e^{-k^2/4} \times\int d^2q\;\sin(\mathbf{k}\wedge\mathbf{q}/2)\Omega^+_{\mathbf{q}}\Omega^-_{\mathbf{q}+\mathbf{k}}. \qquad (68)$$

In the above expression, the momenta are measured in units of the inverse magnetic length l. In the long wavelength approximation (remember that $\Omega^\sigma(\mathbf{q})$ is different from zero only when $|lq| \ll 1$), we have

$$e^{-k^2/4}\sin(\mathbf{k}\wedge\mathbf{q}/2) \approx \mathbf{k}\wedge\mathbf{q}/2 = \hat{z}\cdot(\mathbf{k}\times\mathbf{q})/2,$$

and therefore Eq. (68) can be written as

$$\langle sk|\hat{\rho}_{\mathbf{k}}|sk\rangle = \frac{iN_\phi^2\beta}{2^7\pi^5}\int d^2q\;\hat{z}\cdot\Big((\mathbf{q}+\mathbf{k})\Omega^-_{\mathbf{q}+\mathbf{k}}\Big)\times\Big(\mathbf{q}\Omega^+_{\mathbf{q}}\Big) = -\frac{N_\phi^2\beta}{2^5\pi^3}\epsilon^{\alpha\beta}\int d^2r\;e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{z}\cdot\big(\nabla n^\alpha\times\nabla n^\beta\big) \qquad (69)$$

In the second step, we use the fact that $\mathbf{\Omega}(\mathbf{r}) = \hat{z}\times\mathbf{n}(\mathbf{r})$, hence $\Omega^x = -n^y$, $\Omega^y = n^x$ and $\Omega^z = 0$. From equation (69), the Fourier transform of $\hat{\rho}_{\mathbf{k}}$ is given by

$$\hat{\rho}(\mathbf{r}) \equiv \langle sk|\hat{\rho}_{\mathbf{r}}|sk\rangle = -\frac{N_\phi^2\beta}{4\pi^2}\frac{1}{8\pi}\epsilon^{\alpha\beta}\hat{z}\cdot\big(\nabla n^\alpha\times\nabla n^\beta\big) \qquad (70)$$

As we have assumed that $\mathbf{\Omega}(\mathbf{r})$ corresponds to a small rotation angle of the local spin, $\nabla n^z \approx 0$ and $n^z \approx 1$. Therefore, we can write

$$\epsilon_{\alpha\beta}\hat{z}\cdot\big(\nabla n^\alpha\times\nabla n^\beta\big) \approx \epsilon_{\alpha\beta}n^z\hat{z}\cdot\big(\nabla n^\alpha\times\nabla n^\beta\big) \approx \epsilon^{\alpha\beta}\mathbf{n}\cdot(\partial_\alpha\mathbf{n}\times\partial_\beta\mathbf{n}),$$

as it was done in Ref. 18. Moreover, if we choose the value of the constant $\beta = 4\pi^2/N_\phi^2$, Eq. (70) is in agreement with the definition of the topological charge density (63).

Following the same approximations, we will calculate the average value of the energy of the state $|sk\rangle$ for the

interacting bosonic Hamiltonian (50). The average value of the quadratic term of the Hamiltonian (50) is given by

$$\langle sk|\mathcal{H}_0|sk\rangle \;=\; \frac{l^2}{4(2\pi)^3}\int d^2q\; w_{\mathbf{q}}\Omega^+_{-\mathbf{q}}\Omega^-_{\mathbf{q}}. \qquad (71)$$

Considering the long wavelength limit of the dispersion relation (51), i.e,

$$w_{\mathbf{q}} \approx g + \frac{1}{4}\epsilon_B|lq|^2,$$

Eq. (71) can be written as

$$\langle sk|\mathcal{H}_0|sk\rangle \;\approx\; \langle\mathcal{H}_{\mathcal{Z}}\rangle + \langle\mathcal{H}_{\mathcal{G}}\rangle, \qquad (72)$$

where the Zeeman term is given by

$$\langle\mathcal{H}_{\mathcal{Z}}\rangle \;=\; \frac{l^2}{4(2\pi)^3}g\int d^2q\;\Omega^+_{-\mathbf{q}}\Omega^-_{\mathbf{q}},$$

and the gradient term by

$$\langle\mathcal{H}_{\mathcal{G}}\rangle \;=\; \frac{\epsilon_B}{4}\frac{l^2}{4(2\pi)^3}\int d^2q\;(lq)^2\Omega^+_{-\mathbf{q}}\Omega^-_{\mathbf{q}}.$$

Here, the constant ϵ_B is defined in Appendix A.

Rescaling the momenta by l^{-1} and calculating the Fourier transform, we can show that the Zeeman term can be written as

$$\begin{aligned}\langle sk|\mathcal{H}_{\mathcal{Z}}|sk\rangle \;&=\; \frac{1}{2}g^*\mu_B B\frac{1}{4\pi}\int d^2r\;(n^x)^2 + (n^y)^2\\ &\approx\; -\frac{1}{2}g^*\mu_B\frac{1}{2\pi}\int d^2r\;\mathbf{n}\cdot(\hat{z}B)\\ &\quad+\frac{1}{2}g^*\mu_B B\frac{1}{2\pi}N_\phi. \end{aligned}\qquad (73)$$

In the second step above, we use the identity

$$|\mathbf{n}-\hat{z}|^2 = (n^x)^2 + (n^y)^2 + (n^z-1)^2 = 2 - 2\mathbf{n}\cdot\hat{z}$$

and the fact that, within our approximation, $|n^z-1| \ll 1$. On the other hand, the gradient term assumes the form

$$\begin{aligned}\langle sk|\mathcal{H}_0|sk\rangle \;&=\; \frac{1}{2}\rho_S\int d^2r\;\left[(\nabla\Omega^x)^2 + (\nabla\Omega^y)^2\right]\\ &\approx\; \frac{1}{2}\rho_S\int d^2r\;\left[\nabla\mathbf{n}(\mathbf{r})\right]^2, \end{aligned}\qquad (74)$$

where ρ_S is the spin stiffness as defined in Eq. (62).

Finally, for the interacting term of the Hamiltonian (50), we have

$$\begin{aligned}\langle\mathcal{H}_{int}\rangle \;&=\; \frac{1}{2\mathcal{A}N_\phi^2}\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}V(\mathbf{k})e^{-|lk|^2/2}\sin(\mathbf{k}\wedge\mathbf{p}/2)\\ &\quad\times\sin(\mathbf{k}\wedge\mathbf{q}/2)\,\Omega^-_{\mathbf{k}+\mathbf{p}}\Omega^-_{\mathbf{q}-\mathbf{k}}\Omega^+_{-\mathbf{p}}\Omega^-_{-\mathbf{q}}\\ &\approx\; -\frac{1}{2l^2(8\pi^2)^2}\int d^2k\;V(k/l)\\ &\quad\times\int d^2p\;\hat{z}\cdot\Big((\mathbf{p}+\mathbf{k})\Omega^-_{\mathbf{p}+\mathbf{k}})\times(\mathbf{p}\Omega^+_{\mathbf{p}})\Big)\\ &\quad\times\int d^2q\;\hat{z}\cdot\Big((\mathbf{q}-\mathbf{k})\Omega^-_{\mathbf{q}-\mathbf{k}})\times(\mathbf{q}\Omega^+_{\mathbf{q}})\Big)\\ &=\; \frac{1}{2l}\int d^2r\,d^2r'\;V(\mathbf{r}-\mathbf{r}')q(\mathbf{r})q(\mathbf{r}'), \end{aligned}\qquad (75)$$

where $V(\mathbf{r}-\mathbf{r}') = e^2/(\epsilon|\mathbf{r}-\mathbf{r}'|)$ is the Coulomb potential, $q(\mathbf{r})$ is the topological charge density as defined in [Eq. (70)] and the vector $\mathbf{r}$ is measured in units of l.

From Eqs. (73), (74), and (75), we can conclude that the average value of the energy of the state $|sk\rangle$ is given by

$$\begin{aligned}\langle sk|\mathcal{H}|sk\rangle \;&=\; \frac{1}{2}\rho_S^0\int d^2r\;\left[\nabla\mathbf{n}(\mathbf{r})\right]^2 + \frac{1}{2}g^*\mu_B B\frac{1}{2\pi}N_\phi\\ &\quad-\frac{1}{2}g^*\mu_B\frac{1}{2\pi}\int d^2r\;\mathbf{n}\cdot\mathbf{B}\\ &\quad+\frac{e^2}{2\epsilon l}\int d^2r\,d^2r'\;\frac{q(\mathbf{r})q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned}\qquad (76)$$

Notice that Eq. (76) is equal to the energy functional derived from the Lagrangean density (62) for a static configuration of the vector field $\mathbf{n}(\mathbf{r})$.

It is important to mention that the choice of the constant β, based on the fact that Eq. (70) should be equal to the topological charge density (63) related to the vector field $\mathbf{n}(\mathbf{r})$, gave us the correct values of the coefficients of the Zeeman (73), gradient (74), and interacting (75) terms.

In addition to the approach discussed in Ref. 18, a different method to derive an effective field theory for the quantum Hall skyrmion is presented in.[39,40,41]

V. SUMMARY

We developed a bosonization approach for the 2DEG at $\nu = 1$ using the fact that, at some level of approximation, the elementary neutral excitations of the system can be treated as bosons. The Hamiltonian of the 2DEG at $\nu = 1$, the electron density, and spin density operators were bosonized. We showed that the bosonic representation of the spin density operators is analogous to the one considered by Dyson to study the ferromagnetic Heisenberg model. Furthermore, we showed that the developed

bosonization method is closely related to the LLL projection formalism developed by Girvin and Jach.

The method was applied to study the interacting two-dimensional electron gas at $\nu = 1$. The Hamiltonian of the fermionic system was recast in an interacting two-dimensional boson model. We showed that the dispersion relation of the bosons is equal to the previous diagrammatic calculations of Kallin and Halperin. Within our bosonization approach, we can go beyond the latter results as we also found an interaction between the bosons.

Finally, we showed that the derived interacting bosonic model can describe the quasiparticle-quasihole pair excitation of the 2DEG at $\nu = 1$. On one hand, we showed that the interaction between the bosons accounts for the formation of bound states of two bosons. Our results agree with the previously developed numerical approach of Rezayi's, who studied the quasiparticles of the system. On the other hand, we showed that the semiclassical limit of the interacting bosonic Hamiltonian recovers the energy functional derived from the model suggested by Sonhdi *et al.* to describe the quantum Hall skyrmion.

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APPENDIX A: ENERGY AND LENGTH SCALES FOR THE 2DEG

In the table below, we show the cyclotron $\hbar w_c$, Zeeman g and Coulomb ϵ_c energies and the value of the constants ϵ_B and ρ_s, in Kelvin, as a function of the magnetic field B. The magnetic length $l = \sqrt{\hbar c/(eB)} = 256/\sqrt{B}$ is measured in angstroms and the magnetic field B in Tesla.

Energy scales		(K)
$\hbar w_c$	$\hbar eB/(m^* c)$	$18.78B$
g	$g^* \mu_B B$	$0.33B$
ϵ_c	$e^2/(\epsilon l)$	$50.40\sqrt{B}$
ϵ_B	$\sqrt{\pi/2}\epsilon_c$	$63.16\sqrt{B}$
ρ_s	$\epsilon_c/(16\sqrt{2\pi})$	$1.25\sqrt{B}$

The electron effective mass in the GaAs quantum well is $m^* = 0.07 m_e$, where m_e is the electron mass and the dielectric constant of the semiconductor is $\epsilon \approx 13$.

APPENDIX B: CHARGED PARTICLE IN A PERPENDICULAR MAGNETIC FIELD

Let's consider an electron moving in the $x - y$ plane under the action of a constant perpendicular magnetic field $\mathbf{B} = B_0 \hat{z}$. The Hamiltonian of the system is given by

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 \tag{B1}$$

where $\mathbf{p}$ is the momentum canonically conjugated to $\mathbf{r}$ and $\mathbf{A}$ is the vector potential. In the symmetric gauge,

$$\mathbf{A}(\mathbf{r}) = -\frac{1}{2}\mathbf{r} \times \mathbf{B} = -\frac{1}{2}B_0(y\hat{x} - x\hat{y}).$$

Classically, the electron moves in a circular orbit with angular frequency $w_c = eB/mc$ (cyclotron frequency). In this case, the modulus of the particle velocity

$$\mathbf{v} = \frac{1}{m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})\right)$$

and the position of the center of the cyclotron orbit (guiding center)

$$\mathbf{R_0} = \mathbf{r} + \frac{\hat{z} \times \mathbf{v}}{w_c}$$

are constants of motion.

Defining the complex variables $V = v_x + iv_y$, $P = p_x + ip_y$, $Z = x + iy$, and $Z_0 = R_{0x} + iR_{0y}$, in the symmetric gauge, the constants of motion can be written as

$$\begin{aligned} V &= \frac{1}{m}P + \frac{1}{2}iw_c Z, \\ Z_0 &= Z + \frac{i}{w_c}V. \end{aligned}$$

Now, if we apply the canonical quantization rule for the canonical conjugate variables $\mathbf{r}$ and $\mathbf{p}$, the commutation relations between V and Z_0 are given by

$$\begin{aligned} [V, V^\dagger] &= -\frac{2\hbar w_c}{m} = -2l^2 w_c^2, \\ [Z_0, Z_0^\dagger] &= \frac{2\hbar}{m w_c} = 2l^2, \\ [V^\dagger, Z_0] &= [V, Z_0] = 0, \end{aligned}$$

where l is the magnetic length. Introducing two independent ladder operators d and g, such that $[d, d^\dagger] = [g, g^\dagger] = 1$ and $[d, g] = [d, g^\dagger] = 0$, we can write

$$\begin{aligned} V &= -i\sqrt{2} l w_c d^\dagger, \\ Z_0 &= \sqrt{2} l g. \end{aligned} \tag{B2}$$

It is easy to prove that the operators V and Z_0 defined as in Eq. (B2) satisfy the above commutation relations.

Therefore the Hamiltonian (B1) can be written as

$$\mathcal{H}_0 = \hbar w_c(d^\dagger d + \frac{1}{2}), \tag{B3}$$

whose energy eigenvalues (*Landau levels*) are given by

$$E_{n,m} = \hbar w_c(n + \frac{1}{2}) \tag{B4}$$

and the energy eigenvectors by

$$\begin{aligned} |n\, m\rangle &= \frac{(d^\dagger)^n(g^\dagger)^m}{\sqrt{n!m!}}|0\, 0\rangle, \\ \langle \mathbf{r}|0,0\rangle &= \frac{1}{\sqrt{2\pi l^2}}e^{-r^2/4l^2}, \\ \langle \mathbf{r}|n\, m\rangle &= \frac{1}{\sqrt{2\pi l^2}}e^{\frac{-|r|^2}{4l^2}}G_{m+n,n}(\frac{ir}{l}). \end{aligned} \tag{B5}$$

Here the function $G_{m+n,n}(x)$ is defined in Appendix C.

Semiclassically, the state $|n\, m\rangle$ can be seen as an electron in a cyclotron orbit with radius equal to $l\sqrt{2n+1}$ and the center located at a distance $l\sqrt{2m+1}$ from the origin of the coordinate system.

A detailed analysis of this problem is presented in Ref. 10. Our formalism is similar to the one presented in this reference with the replacements $g \to b^\dagger$ and $d^\dagger \to -ia$.

APPENDIX C: THE $G_{m,m'}(lq)$ FUNCTION PROPERTIES

We want to calculate the matrix element of the operator $e^{-i\mathbf{q}\cdot\mathbf{r}}$ in the Landau level basis. Writing $q = q_x + iq_y$ and $r = x + iy$, we can expand the latter in terms of the ladder operators d and g defined in Appendix B [see Eq. (B2)],

$$r = Z = Z_0 - \frac{i}{w_c}V = \sqrt{2}l(g - d^\dagger), \tag{C1}$$

Therefore the matrix element becomes

$$\begin{aligned} \langle n\, m|e^{-i\mathbf{q}\cdot\mathbf{r}}|n'\, m'\rangle &= \langle n\, m|\exp\left(-i(qr^* + q^*r)/2\right)|n'\, m'\rangle \\ &= \langle n\, m|\exp\left[-il\left((qg + q^*g^\dagger) - (q^*d + qd^\dagger)\right)/\sqrt{2}\right]|n'\, m'\rangle. \end{aligned} \tag{C2}$$

Since the ladder operators d and g are related only to the Landau levels and the guiding centers respectively, we can use the properties of these operators to write the above matrix element as a product

$$\langle n\, m|e^{-i\mathbf{q}\cdot\mathbf{r}}|n'\, m'\rangle = \exp(-|lq|^2/2)G_{m,m'}(lq)G_{n,n'}(-lq^*), \tag{C3}$$

where the functions $G_{m,m'}(lq)$ and $G_{n,n'}(-lq^*)$ are defined as

$$\begin{aligned} G_{m,m'}(lq) &\equiv \langle m|\exp(-ilqg^\dagger/\sqrt{2})\exp(-ilq^*g/\sqrt{2})|m'\rangle, \end{aligned} \tag{C4}$$

$$G_{n,n'}(-lq^*) \equiv \langle n|\exp(ilq^*d/\sqrt{2})\exp(ilqd^\dagger/\sqrt{2})|n'\rangle.$$

Now, if we take $n = n' = 0$ in Eq. (C3), we have the matrix element of the operator $e^{-i\mathbf{q}\cdot\mathbf{r}}$ in the lowest Landau level basis

$$\langle m|e^{-i\mathbf{q}\cdot\mathbf{r}}|m'\rangle = \exp(-|lq|^2/2)G_{m,m'}(lq). \tag{C5}$$

In the LLL projection formalism, the above expression corresponds to the matrix element of projected operator $e^{-i\mathbf{q}\cdot\mathbf{r}}$ [compare Eq. (C5) with Eq. (25.1.11) of Ref. 10).

Using the properties of the ladder operators, it is possible to show that the function $G_{m,m'}(lq)$ can be written as a linear combination of the generalized Laguerre polynomials $L_{m'}^{m-m'}(|lq|^2/2)$, i.e.,

$$\begin{aligned} G_{m,m'}(lq) &= \theta(m'-m)\sqrt{\frac{m!}{m'!}}\left(\frac{-ilq^*}{\sqrt{2}}\right)^{m'-m}L_m^{m'-m}\left(\frac{|lq|^2}{2}\right) \\ &+\theta(m-m')\sqrt{\frac{m'!}{m!}}\left(\frac{-ilq}{\sqrt{2}}\right)^{m-m'}L_{m'}^{m-m'}\left(\frac{|lq|^2}{2}\right). \end{aligned} \tag{C6}$$

From expressions (C4) and (C6) we can prove the following properties of the function $G_{m,m'}(lq)$,

(i) relations between the function and its complex conjugate:

$$\begin{aligned} G_{m,m'}(lq) &= G^*_{m,m'}(-lq^*) = G^*_{m',m}(-lq) = G_{m',m}(lq^*) \\ G_{m,m'}(ilq) &= G^*_{m,m'}(ilq^*) = G^*_{m',m}(-ilq^*) = (-i)^{m-m'} G_{m',m}(lq^*). \end{aligned} \tag{C7}$$

(ii) The Fourier transform of the product of two functions:

$$\begin{aligned} e^{\frac{-|lq|^2}{2}} G_{m,m'}(lq) G_{n,n'}(-lq^*) &= \int d^2r\ e^{-i\mathbf{q}\cdot\mathbf{r}} \langle n', m'|\mathbf{r}\rangle\langle\mathbf{r}|n,m\rangle \\ &= \frac{1}{2\pi l^2}\int d^2r\ e^{-i\mathbf{q}\cdot\mathbf{r}} e^{\frac{-|r|^2}{2l^2}} G_{n+m,n}\left(\frac{ir}{l}\right) G_{n',m'+n'}\left(\frac{-ir}{l}\right). \end{aligned} \tag{C8}$$

(iii) The sum of the product of two functions: as the Landau level basis $|n,m\rangle$ is a complete basis, we have

$$\begin{aligned} \sum_l G_{m,l}(lq) G_{l,m'}(lk) &= \sum_l \langle m|\exp(-ilqb^\dagger/\sqrt{2})\exp(-ilq^*b/\sqrt{2})|l\rangle\langle l|\exp(-ilkb^\dagger/\sqrt{2})\exp(-ilk^*b/\sqrt{2})|m'\rangle \\ &= \exp\left(\frac{-l^2q^*k}{2}\right) G_{m,m'}(lq+lk). \end{aligned} \tag{C9}$$

(iv) Orthogonality relation: using the orthogonality relations of the generalized Laguerre polynomials, we can show that

$$\int d^2k\ e^{-|lk|^2/2} G_{m,m'}(-lk^*) G_{n,n'}(lk) = \frac{2\pi}{l^2}\delta_{m,n}\delta_{m',n'}, \tag{C10}$$

and changing the integral over momenta by a sum,

$$\sum_{\mathbf{k}} e^{-|lk|^2/2} G_{m,m'}(-lk^*) G_{n,n'}(lk) = N_\phi \delta_{m,n}\delta_{m',n'}. \tag{C11}$$

(v) The trace:

$$\begin{aligned} \sum_m G_{m,m}(lq) &= \frac{e^{|lq|^2/2}}{2\pi l^2}\sum_m \int d^2r\ e^{-i\mathbf{q}\cdot\mathbf{r}} e^{\frac{-|r|^2}{2l^2}} G_{0,m}(\frac{-ir}{l}) G_{m,0}(\frac{ir}{l}) \\ &= \frac{e^{|lq|^2/2}}{2\pi l^2}\int d^2r\ e^{-i\mathbf{q}\cdot\mathbf{r}} e^{\frac{-|r|^2}{2l^2}} e^{\frac{|r|^2}{2l^2}} \underbrace{G_{0,0}(0)}_{1} = N_\phi \delta(\mathbf{q}). \end{aligned} \tag{C12}$$

APPENDIX D: THE COMMUTATOR $[S^+_{\mathbf{q}}, S^-_{\mathbf{q}'}]$

If we consider the expressions of the spin operators $S^+_{\mathbf{q}}$ and $S^-_{\mathbf{q}}$ in terms of the fermionic annihilation and creation operators [Eq. (12) and (13)], we have

$$\begin{aligned} [S^+_{\mathbf{q}}, S^-_{\mathbf{q}'}] &= e^{-|lq|^2/2-|lq'|^2/2}\sum_{m,m',n,n'} G_{m,m'}(lq) G_{n,n'}(lq')[c^\dagger_{m\uparrow}c_{m'\downarrow}, c^\dagger_{n\downarrow}c_{n'\uparrow}] \\ &= e^{-|lq|^2/2-|lq'|^2/2}\left(\sum_{m,n,n'} G_{m,n}(lq)G_{n,n'}(lq')c^\dagger_{m\uparrow}c_{n'\uparrow} - \sum_{m,m',n} G_{n,m}(lq')G_{m,m'}(lq)c^\dagger_{n\downarrow}c_{m'\downarrow}\right) \\ &= e^{-(|lq|^2/2+|lq'|^2/2)}\left(e^{-l^2q^*q'/2}\sum_{m,n} G_{m,n}(lq+lq')c^\dagger_{m\uparrow}c_{n\uparrow} - e^{-l^2q'^*q/2}\sum_{m,n} G_{m,n}(lq'+lq)c^\dagger_{m\downarrow}c_{n\downarrow}\right) \\ &= e^{l^2qq'^*/2}e^{-|lq+lq'|^2/2}\sum_{m,n} G_{m,n}(lq+lq')c^\dagger_{m\uparrow}c_{n\uparrow} - e^{l^2q'q^*/2}e^{-|lq+lq'|^2/2}\sum_{m,n} G_{m,n}(lq'+lq)c^\dagger_{m\downarrow}c_{n\downarrow}. \end{aligned} \tag{D1}$$

Now, if we compare the above result with the expressions of the electron density operators $\hat{\rho}_\sigma(\mathbf{q})$ [Eq. (11)], we can conclude that

$$[S^+_{\mathbf{q}}, S^-_{\mathbf{q}'}] = e^{l^2 q q'^*/2}\hat{\rho}_\uparrow(\mathbf{q}+\mathbf{q}') - e^{l^2 q' q^*/2}\hat{\rho}_\downarrow(\mathbf{q}+\mathbf{q}'). \tag{D2}$$

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